On a new convergence class in k-bounded sober spaces

Hadrian Andradi and Weng Kin Ho
MME NIE-NTU, Singapore

November 27, 2015

Some Problems

1. Collatz Conjecture
   Given $a_0 \in \mathbb{Z}^+$. For all $i > 0$, define
   $$a_i = \begin{cases} \frac{a_{i-1}}{2} & \text{if } a_{i-1} \text{ is even} \\ 3a_{i-1} + 1 & \text{if } a_{i-1} \text{ is odd} \end{cases}$$
   For any initial input $a_0 \in \mathbb{Z}^+$, the Collatz sequence will eventually reach 1.

Some Problems

1. Collatz Conjecture
2. The Lonely Runner Conjecture
   Consider k runners on a circular track of unit length. At $t = 0$, all runners are at the same position and start to run; the runners' speeds are pairwise distinct.
   A runner is said to be lonely at time t if he is at a distance of at least $\frac{1}{k}$ from every other runner at time t.
   Each runner is lonely at some time.

Some Problems

1. Collatz Conjecture
2. The Lonely Runner Conjecture
3. Hadwiger Conjecture in Graph Theory
   Does every graph with chromatic number k have K_k as a minor?

Some Problems

1. Collatz Conjecture
2. The Lonely Runner Conjecture
3. Hadwiger Conjecture in Graph Theory
4. Goldbach's Conjecture
5. The Twin Prime Conjecture
6. Riemann Hypothesis

What to do?

What to do?

Find a reasonable problem or pose your own problem.

What to do?

Find a reasonable problem or pose your own problem.

Question some possible questions.

What to do?

Find a reasonable problem or pose your own problem.

Question some possible questions.

Be sensitive about established results.

What to do?

Find a reasonable problem or pose your own problem.

Question some possible questions.

Be sensitive about established results.

Stay focus.

Fermat's Last Theorem

$$a + b = c$$

$$\downarrow$$

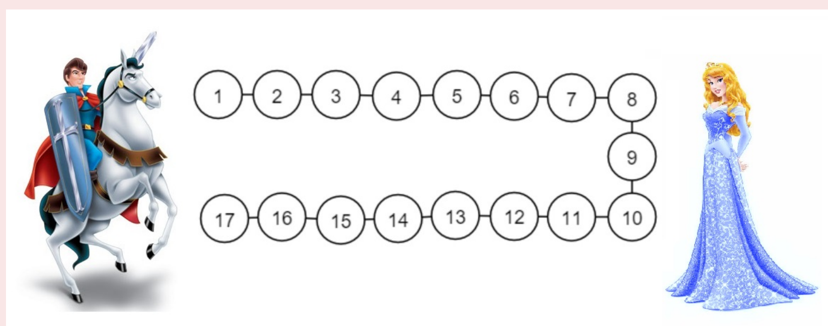
$$a^2 + b^2 = c^2$$

$$\downarrow$$

$$a^n + b^n = c^n$$

for $n \geq 3$

The Princess Problem

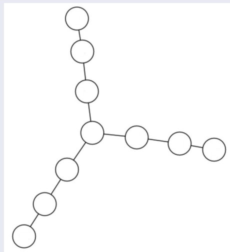


What is the strategy?

The princess is moved from one room to another adjacent room each morning. The prince can only open one door each afternoon. How to meet the princess in 30 days?

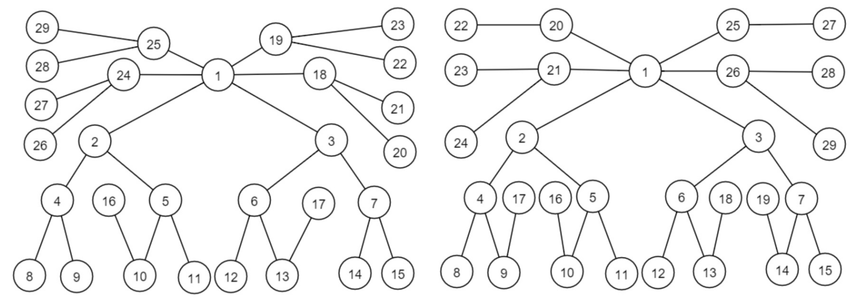
Theorem [Britnell and Wildon(2013)]

Let G be a tree representing the shape of the arrangement of the rooms. If G does not contain any subgraph isomorphic to the following graph



then the prince can guarantee to find the princess. In any other cases, the princess can avoid the princess indefinitely.

Which one is solvable?



A T_0 space X can be considered as a poset whose order is the specialization order induced by the topology on X.

A T_0 space X can be considered as a poset whose order is the specialization order induced by the topology on X.

It was pointed out that several results in domain theory can be upgraded from the contexts of posets to T_0 spaces.

1. The topological technique of dcpo-completion of posets [Zhao and Fan(2007)] can be upgraded to yield D-completion of T_0 spaces [Keimel and Lawson(2009)].

$$\begin{tikzcd}
\text{Poset} \arrow[d, "\text{dcpo-completion}"'] & & T_0 \text{ space} \arrow[d, "D\text{-completion}"] \\
\text{dcpo} & \arrow[r, phantom, ""] \xrightarrow{\text{Topological parallel}} & \text{d-space}
\end{tikzcd}$$

2. Topological version of Rudin's Lemma [Heckmann and Keimel(2013)].

If $\mathcal{F}$ is a $\sqsubseteq$-directed family of nonempty finite subsets of a poset P then there exists a directed subset $D \subseteq \bigcup \mathcal{F}$ that intersects each member of $\mathcal{F}$ nonemptily.

$\Big\downarrow$ Topological parallel

For every $\sqsubseteq$-directed collection $\mathcal{A}$ of nonempty compact saturated subsets of a space X, there is an irreducible (closed) subset A meeting all members of $\mathcal{A}$.

3. The irreducibly derived topology [Zhao and Ho(2015)] which in some sense is a topological analogue of Scott topology.

Let P be a poset endowed with the Alexandroff topology. A subset U of P is Scott open iff U is open and U is inaccessible by supremum of directed sets.

$\Big\downarrow$ Topological parallel

Let X be a T_0 space. A subset U of X is SI-open iff U is open and U is inaccessible by supremum of irreducible sets.

Our goal is to find a topological parallel of the following theorem given by B. Zhao and D. Zhao:

"The lim-inf convergence class on a poset P is topological if and only if P is a continuous poset."

Result of B. Zhao and D .Zhao [Zhao and Zhao(2005)]:
The following are equivalent for a poset P:

1. The lim-inf convergence class on P is topological.
2. P is continuous.

$\Big\downarrow$ Topological parallel??

The following are equivalent for a space X:

1. The net convergence class $\mathcal{I}$ on X is topological.
2. X is Irr-continuous.

Directed set

A nonempty subset D of a poset $(P, \leq)$ is directed if for any $d_1, d_2 \in D$ there exists $d \in D$ such that $d_1, d_2 \leq d$.

$$\mathrm{Dir}(P) := \{D \subseteq P \mid D \text{ is directed}\}$$

Directed set

A nonempty subset D of a poset $(P, \leq)$ is directed if for any $d_1, d_2 \in D$ there exists $d \in D$ such that $d_1, d_2 \leq d$.

$$\mathrm{Dir}(P) := \{D \subseteq P \mid D \text{ is directed}\}$$

Irreducible set

A nonempty subset E of a T_0 space (X, τ) is irreducible if for any closed sets A_1 and A_2, whenever $E \subseteq A_1 \cup A_2$, it holds that $E \subseteq A_1$ or $E \subseteq A_2$.

$$\mathrm{Irr}_\tau(X) := \{E \subseteq X \mid E \text{ is irreducible}\}$$

Directed set

A nonempty subset D of a poset $(P, \leq)$ is directed if for any $d_1, d_2 \in D$ there exists $d \in D$ such that $d_1, d_2 \leq d$.

$$\mathrm{Dir}(P) := \{D \subseteq P \mid D \text{ is directed}\}$$

Irreducible set

A nonempty subset E of a T_0 space (X, τ) is irreducible if for any closed sets A_1 and A_2, whenever $E \subseteq A_1 \cup A_2$, it holds that $E \subseteq A_1$ or $E \subseteq A_2$.

$$\mathrm{Irr}_\tau(X) := \{E \subseteq X \mid E \text{ is irreducible}\}$$

Fact

1. In a T_0 space X, every directed space is irreducible.
2. In a poset P endowed with Alexandroff topology, it holds that $\mathrm{Irr}(P) = \mathrm{Dir}(P)$.

Scott topology

Let P be a poset endowed with the Alexandroff topology and $U \subseteq P$. Define $U \in \sigma(P)$ if

1. U is Alexandroff open, and
2. for every $D \in \mathrm{Irr}^+(P)$, $\bigvee D \in U$ implies $D \cap U \neq \varnothing$.

The collection of $\sigma(P)$ forms a topology on P, called Scott topology.

Scott topology

Let P be a poset endowed with the Alexandroff topology and $U \subseteq P$. Define $U \in \sigma(P)$ if

1. U is Alexandroff open, and
2. for every $D \in \mathrm{Irr}^+(P)$, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$.

The collection of $\sigma(P)$ forms a topology on P, called Scott topology.

Irreducibly derived topology

Let (X, τ) be a T_0 space and $U \subseteq X$. Define $U \in \tau_{SI}$ if

1. $U \in \tau$, and
2. for every $E \in \mathrm{Irr}^+_\tau(X)$, $\bigvee E \in U$ implies $E \cap U \neq \emptyset$.

The collection of τ_{SI} forms a topology on X. We denote the space (X, τ_{SI}) by $SI(X)$ and call τ_{SI} irreducibly derived topology.

Proposition [Zhao and Ho(2015)]

For any T_0 space X, the following hold:

1. For any $x \in X$, $\mathrm{cl}_X(\{x\}) = \mathrm{cl}_{SI(X)}(\{x\})$
2. A closed subset C of X is closed in $SI(X)$ if and only if for every $E \in \mathrm{Irr}^+_\tau(X)$, $E \subseteq C$ implies $\sup E \in C$.
3. A subset U of X is clopen in X if and only if it is clopen in $SI(X)$.
4. X is connected if and only if $SI(X)$ is connected.

k-bounded sober space

A topological space is k-bounded sober if every closed set $F \in \mathrm{Irr}^+(X)$ is the closure of a unique singleton.

Compare

1. Sober space: if every closed set $F \in \mathrm{Irr}(X)$ is the closure of a unique singleton.
2. Bounded sober space: if every closed bounded above set $F \in \mathrm{Irr}^+(X)$ is the closure of a unique singleton.

Example

1. [Zhao and Ho(2015)] The set $\mathbb{Q}$ equipped with the upper topology is k-bounded sober but not bounded sober.
2. [Zhao and Fan(2007)] The Scott space of real numbers $\mathbb{R}$ is bounded sober but not sober.

Definition

A topological space (X, τ) is said to satisfy the SI^{∞} property if $\tau = \tau_{SI}$.

Theorem [Zhao and Ho(2015)]

A T_0 space X is k-bounded sober if and only if X satisfies the SI^{∞} property.

Way-below relation

Given $p, q \in P$, the way-below relation on P is defined as follow

$$p \ll q \iff \forall D \in \mathrm{Irr}^{+}(P).\ (\bigvee D \geq q) \implies (D \cap \uparrow p \neq \varnothing).$$

$$\downarrow p := \{q \in X \mid q \ll p\}$$

Way-below relation

Given $p, q \in P$, the way-below relation on P is defined as follow

$$p \ll q \iff \forall D \in \mathrm{Irr}^{+}(P).\ (\bigvee D \geq q) \implies (D \cap \uparrow p \neq \varnothing).$$

$$\downarrow p := \{q \in X \mid q \ll p\}$$

Irr-way-below relation

Given $x, y \in X$, the Irr-way-below relation on X is defined as follow

$$x \ll_{\mathrm{Irr}} y \iff \forall E \in \mathrm{Irr}^{+}(X).\ (\bigvee E \geq y) \implies (E \cap \uparrow x \neq \varnothing).$$

$$\downarrow_{\mathrm{Irr}} x := \{y \in X \mid y \ll_{\mathrm{Irr}} x\}$$

Proposition

In a poset P the following hold for all p, q, r and $s \in X$:

1. $p \ll q$ implies $p \leq q$.
2. $p \leq q \ll r \leq s$ implies $p \ll s$.

Proposition

In a poset P the following hold for all p, q, r and $s \in X$:

1. $p \ll q$ implies $p \leq q$.
2. $p \leq q \ll r \leq s$ implies $p \ll s$.

Proposition

In a space X the following hold for all u, x, y and $z \in X$:

1. $x \ll_{\mathrm{Irr}} y$ implies $x \leq y$.
2. $u \leq x \ll_{\mathrm{Irr}} y \leq z$ implies $u \ll_{\mathrm{Irr}} z$.

Continuous poset

A poset $(P, \leq)$ is said to be continuous if for every $p \in P$ the following hold:

1. $\twoheaddownarrow p$ is irreducible in P endowed with Alexandroff topology and
2. $p = \bigvee \twoheaddownarrow p$.

Continuous poset

A poset $(P, \leq)$ is said to be continuous if for every $p \in P$ the following hold:

1. $\twoheaddownarrow p$ is irreducible in P endowed with Alexandroff topology and
2. $p = \bigvee \twoheaddownarrow p$.

Irr-continuous space

A topological space (X, τ) is said to be Irr-continuous if for every $x \in X$ the following hold:

1. $\twoheaddownarrow_{\mathrm{Irr}} x$ is irreducible in (X, τ) and
2. $x = \bigvee \twoheaddownarrow_{\mathrm{Irr}} x$.

lim-inf convergence

Let P be a poset endowed with Alexandroff topology. A net $(x_i)_{i \in I}$ in P is said to lim-inf converge to $y \in P$ if there exists $D \in \mathrm{Irr}^{+}(P)$ such that $\bigvee D \geq y$ and for each $d \in D$,$(x_i)_{i \in I}$ is eventually greater than d.

lim-inf convergence

Let P be a poset endowed with Alexandroff topology. A net $(x_i)_{i\in I}$ in P is said to lim-inf converge to $y \in P$ if there exists $D \in \mathrm{Irr}^+(P)$ such that $\bigvee D \geq y$ and for each $d \in D$,$(x_i)_{i\in I}$ is eventually greater than d.

Irr-convergence

Let (X,τ) be a topological space. A net $(x_i)_{i\in I}$ in (X,τ) is said to Irr-converge to $y \in X$ if there exists $E \in \mathrm{Irr}^+(X)$ such that $\bigvee E \geq_\tau y$ and for each $e \in E$,$(x_i)_{i\in I}$ is eventually greater than e.

Definition

For a T_0 space X, define the class convergence $\mathcal{I}$ as follows:

$$\mathcal{I} = \{((x_i)_{i\in I}, y) \mid (x_i)_{i\in I}\ \mathrm{Irr}\text{-converge to } y\}$$

Kelley's characterisation for topological converngence class

A convergence class $\mathcal{S}$ is topological if and only if it satisfies the following conditions:

1. (Constants). If $(x_i)_{i\in I}$ is a constant net with $x_i = x$ for all i, then $((x_i)_{i\in I}, x) \in \mathcal{S}$.
2. (Subnets). If $((x_i)_{i\in I}, x) \in \mathcal{S}$ and $(y_j)_{j\in J}$ is a subnet of $(x_i)_{i\in I}$, then $\left((y_j)_{j\in J}, x\right) \in \mathcal{S}$.
3. (Divergence). If $((x_i)_{i\in I}, x) \notin \mathcal{S}$, then there exists a subnet $(y_j)_{j\in J}$ of $(x_i)_{i\in I}$ such that for any subnet $(z_k)_{k\in K}$ of $(y_j)_{j\in J}$, $((z_k)_{k\in K}, x) \notin \mathcal{S}$.
4. (Iterated limits). If $((x_i)_{i\in I}, x) \in \mathcal{S}$ and $\left((x_{i,j})_{j\in J(i)}, x_i\right) \in \mathcal{S}$ for all $i \in I$, then $\left((x_{i,f(i)})_{(i,f)\in I\times M}, x\right) \in \mathcal{S}$, where $M := \prod\{J(i) \mid i \in I\}$.

Theorem

Let P be a continuous poset. Then $\ll$ enjoys the interpolation property in that whenever $p \ll r$, there exists $q \in P$ such that

$$p \ll q \ll r.$$

Theorem

Let P be a continuous poset. Then $\ll$ enjoys the interpolation property in that whenever $p \ll r$, there exists $q \in P$ such that

$$p \ll q \ll r.$$

Lemma

Let X be an Irr-continuous space. Then for every $x \in X$ it holds that

$$x = \bigvee \bigcup \{\Downarrow_{\mathrm{Irr}} y \mid y \ll_{\mathrm{Irr}} x\}.$$

Theorem

Let X be an Irr-continuous and k-bounded sober space. Then $\ll_{\mathrm{Irr}}$ enjoys the interpolation property in that whenever $z \ll_{\mathrm{Irr}} x$, there exists $y \in X$ such that

$$z \ll_{\mathrm{Irr}} y \ll_{\mathrm{Irr}} x.$$

Lemma

Let X be a T_0 space.

1. The class $\mathcal{I}$ satisfies the axioms (Constants) and (Subnets).
2. If X is Irr-continuous, then $\mathcal{I}$ satisfies the (Divergence) axiom.
3. If X is Irr-continuous and k-bounded sober, then $\mathcal{I}$ satisfies the (Iterated limits) axiom.

Lemma

For any k-bounded sober space X, if $\mathcal{I}$ satisfies the (Iterated limits) axiom then X is Irr-continuous.

Theorem

The following are equivalent for a poset P:

1. The lim-inf convergence class on P is topological.
2. P is continuous.

Theorem

The following are equivalent for a poset P:

1. The lim-inf convergence class on P is topological.
2. P is continuous.

Main theorem

The following are equivalent for a k-bounded sober space X:

1. The net convergence class $\mathcal{I}$ on X is topological.
2. X is Irr-continuous.

Thank You

References

Britnell, J. and Wildon, M.
Finding a princess in a palace: A pursuit-evasion problem.
The Electronic Journal of Combinatorics, 20(1), 2013.

Heckmann, R. and Keimel, K.
Quasicontinuous Domains and the Smyth Powerdomain.
Electronics Notes in Theoretical Computer Science, 298:215–232, November 2013.

Keimel, K. and Lawson, J. D.
D-completions and d-topology.
Annals of Pure and Applied Logic, 159(3):292–306, June 2009.

Zhao, B. and Zhao, D.
The lim-inf convergence on partially ordered sets.
J. Mathematical Analysis and its applications, 309:701–708, 2005.

Zhao, D. and Fan, T.
Dcpo-completion of posets.
Theoretic Computer Science, 2007.
(Preprint).

Zhao, D. and Ho, W. K.
On topologies defined by irreducible sets.
Journal of Logical and Algebraic Methods in Programming, 84:185–195, 2015.

References

- https://en.wikipedia.org/wiki/Lonely_runner_conjecture
- http://disney.wikia.com/wiki/Prince_Phillip
- http://checkmyworking.com/2011/12/solving-the-princess-on-a-graph-puzzle/
- http://www.india-forums.com/forum_posts.asp?TID=4384984