

Some recent works on
Baire class one functions and
problem posing strategies involved

Zhao Dongsheng

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Continuous functions

$f: \mathbb{R} \rightarrow \mathbb{R}$ is cont. iff

$\forall \varepsilon > 0, \exists \delta: \mathbb{R} \rightarrow \mathbb{R}^+$ such that $\forall x, y \in \mathbb{R}$,

$$|x - y| < \max\{\delta(x), \delta(y)\} \implies |f(x) - f(y)| < \varepsilon. \quad (1)$$

Question: what functions are the f in (1),

$\min\{\delta(x), \delta(y)\}$ is replaced by

$$\min\{\delta(x), \delta(y)\}?$$

H-continuous functions

$f: \mathbb{R} \rightarrow \mathbb{R}$ is H-cont. if

$\forall \varepsilon > 0, \exists \delta: \mathbb{R} \rightarrow \mathbb{R}^+$ s. t. $\forall x, y \in \mathbb{R}$,

$$|x - y| < \min\{\delta(x), \delta(y)\} \implies |f(x) - f(y)| < \varepsilon \quad (2)$$

Question: what are the properties of H-cont. functions.

• If f, g are H-cont, then so are

$$f + g, fg$$

• If $\{f_n\}$ is a sequence of cont. functions and

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \forall x \in \mathbb{R},$$

then f is H-continuous.

Thus every Baire class one function is H-cont.

Question: Is the converse of above also true?

Theorem. (Lee, Tang, Zhao 2007)

Let $f: X \rightarrow Y$ be a mapping between complete metric spaces. Then TFAE:

(1) $\forall \varepsilon > 0, \exists \delta: X \rightarrow \mathbb{R}^+$,

$$d_Y(f(x), f(y)) < \varepsilon \text{ whenever}$$

$$d_X(x, y) < \min\{\delta(x), \delta(y)\}.$$

(2) f is of the first Baire class.

Question: what if $\min\{\delta(x), \delta(y)\}$ is replaced by other two variable functions?

Jachymski & Lindner (2005)

For any $M: (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$, let

$$A_M(x, Y): f: X \rightarrow Y,$$

$$\forall \varepsilon > 0, \exists \delta: X \rightarrow (0, \infty) \text{ s.t. } \forall x_1, x_2 \in X$$

$$d_X(x_1, x_2) < M(\delta(x_1), \delta(x_2)) \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

Proposition. $\forall M: (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$,

\exists symmetric $\hat{M}: (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$ s.t.

$$A_M(x, Y) = A_{\hat{M}}(x, Y).$$

Theorem. Let $M: (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$ be a symmetric and non-decreasing. Then

$$A_M(\mathbb{R}, \mathbb{R}) = B_1(\mathbb{R}, \mathbb{R})$$

$$\text{or } = C(\mathbb{R}, \mathbb{R}).$$

$$\text{Eg. if } M(u, v) = \frac{1}{2}(u, v), \quad A_M(\mathbb{R}, \mathbb{R}) = C(\mathbb{R}, \mathbb{R}) \\ = \sqrt{uv}, \quad A_M(\mathbb{R}, \mathbb{R}) = B_1(\mathbb{R}, \mathbb{R}).$$

Question what if the $\varepsilon > 0$ in (1) is replaced by a function $\varepsilon: \mathbb{R} \rightarrow \mathbb{R}^+$?

K-Continuous functions

$f: \mathbb{R} \rightarrow \mathbb{R}$ is K-continuous if $\forall \varepsilon: \mathbb{R} \rightarrow \mathbb{R}^+$,

$\exists \delta: \mathbb{R} \rightarrow \mathbb{R}^+$ such that $\forall x_1, x_2 \in \mathbb{R}$,

$$|x_1 - x_2| < \min\{\delta(x_1), \delta(x_2)\} \implies |f(x_1) - f(x_2)| < \min\{\varepsilon(f(x_1)), \varepsilon(f(x_2))\}$$

Properties (Zhao D 2007)

(1) If $f, g: \mathbb{R} \rightarrow \mathbb{R}$ are K-Cont, then so is $g \circ f$.

(2) If f, g are inj. K-Cont, then $f+g, fg$ are K-Cont.

(3) If $\forall B1$ $h: \mathbb{R} \rightarrow \mathbb{R}$, $h \circ f$ is B1.

A function f satisfies condition (3) is called a right B1 compositor (Zhao D 2007)

Question: Is every right B1 compositor K-continuous?

Theorem (Fenechos and Emmanuel A. Cabral 2012)

(1) A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is a right B1 compositor iff it is K-continuous.

(2) The sum/product of two K-Cont. functions is K-continuous.

(3) Every B1 function is the uniform limit of a sequence of K-Cont. functions.

Question: what about left B1 compositors?

Theorem (Fenechos and Cabral 2013)

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is a left B1 compositor iff it is continuous.