# A conjecture on chromatic polynomials

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# Outline

1. Chromatic polynomial
2. Tensor product $G \times H$
3. Conjectures
4. Generalization

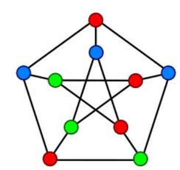


# *Chromatic polynomial*



# Proper coloring

- For a positive integer $k$, a (proper) $k$-coloring of a graph $G$ is a way of assigning $k$ colors to vertices in $G$, one color for each vertex, such that any two adjacent vertices are assigned different colours.



Chromatic number $\chi(G)$ is the minimum $k$ such that $G$ admits a proper $k$-coloring.

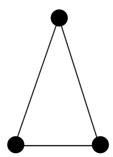
## Brooks Theorem

For any connected graph $G$, if $G$ is not complete nor an odd cycle, then $\chi(G) \leq \Delta(G)$.

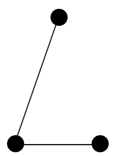
## $P(G,k)$: Counting $k$-colourings

*$P(G,k)$: the number of ways of assigning one color in $\{c_1, \cdots, c_k\}$ to each vertex of G such that any two adjacent vertices are colored differently.*

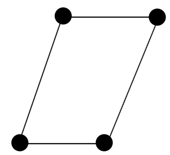
- Examples for $P(G,k)$:



$k(k-1)(k-2)$



$k(k-1)^2$



$(k-1)^4 + (k-1)$



## Chromatic polynomial

$P(G,k)$ is called the chromatic polynomial of $G$.

It was introduced by Birkhoff in 1912 with the hope of proving 4CC.

George David Birkhoff (1884-1944) was one of the most important leaders in American mathematics in his generation.



## $P(G,k)$ is a polynomial

- $P(K_n,k) = k(k-1)\cdots(k-n+1)$ and $P(N_n,k) = k^n$.
- Deletion and contraction formula:

*For any edge e in G,*

$$P(G,k) = P(G\backslash e,k) - P(G/e,k),$$

*where $G\backslash e$ and $G/e$ are the graphs obtained from G by deleting e and contracting e respectively.*

- By the deletion and contraction formula and the fact that $P(N_n,k) = k^n$,

  $P(G,k)$ is a polynomial for any graph $G$.



## Expansion by subgraphs

- It can be proved by the principle of inclusion and exclusion,

*for any simple graph $G = (V,E)$,*

$$P(G,k) = \sum_{A\subseteq E} (-1)^{|A|} k^{c(A)},$$

where $c(A)$ is the number of components in the spanning subgraph $(V,A)$.

- Hence $P(G,k)$ is a polynomial.

## Expansion by partition numbers

- For any positive integer $i$, let $\alpha_i(G)$ be the number of partitions of $V(G)$ into $i$ non-empty independent sets $I_1, I_2, \ldots, I_i$.
- For any graph $G$,

$$P(G,k) = \sum_{i=1}^{|V(G)|} \alpha_i(G)(x)_i,$$

*where* $(x)_i = x(x-1)\cdots(x-i+1)$.

- Given any two graphs $G$ and $H$, $P(G,k)$ and $P(H,k)$ are the same polynomial if and only if

$$\alpha_i(G) = \alpha_i(H), \quad \forall i = 1,2,3,\ldots.$$



## A conclusion

- For any simple graphs $G$ and $H$ of order $n$,

$\alpha_i(G) \geq \alpha_i(H)$ *holds for all* $i \in \{1,2,\ldots,n\}$

$$\Downarrow$$

$P(G,k) \geq P(H,k)$ *for all positive integers* $k$.



*Tensor product $G \times H$*



## Tensor product

- For any simple graphs $G$ and $H$, the tensor product $G \times H$ of $G$ and $H$ is the graph such that its vertex set is

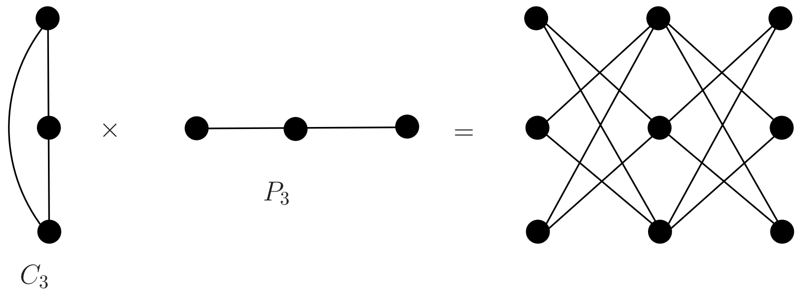
$$\{(u,v) : u \in V(G), v \in V(H)\}$$

and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in $G \times H$ if and only if

  - $u_1$ and $u_2$ are adjacent in $G$; and
  - $v_1$ and $v_2$ are adjacent in $H$.

- Example: $C_3 \times K_2$ is a 6-cycle.

# $C_3 \times P_3$






# Some properties of $G \times K_2$

- $G \times K_2$ is the graph with vertex set $X \cup Y$, where $X = \{(v, 1) : v \in V(G)\}$ and $X = \{(v, 2) : v \in V(G)\}$, and edge set $\{(v_1, 1)(v_2, 2) : v_1v_2 \in E(G)\}$.
- $G \times K_2$ is a bipartite graph, called the bipartite double cover of $G$.
- $G \times K_2$ has $2|V(G)|$ vertices and $2|E(G)|$ edges.
- Thus, $G \times K_2$ and $G \sqcup G$ have the same order and size, where $G \sqcup H$ is the vertex-disjoint union of $G$ and $H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.
- If $G$ is bipartite, then $G \times K_2 = G \sqcup G$.



# On cycles in $G \times K_2$

- For each even cycle $u_1u_2 \cdots u_{2r}u_1$ in $G$, there are two corresponding $2r$-cycles in $G \times K_2$:

$$(u_1, 1)(u_2, 2)(u_3, 1) \cdots (u_{2r}, 2)(u_1, 1);$$

$$(u_1, 2)(u_2, 1)(u_3, 2) \cdots (u_{2r}, 1)(u_1, 2).$$

- For each odd cycle $u_1u_2 \cdots u_{2r+1}u_1$ in $G$, there is a corresponding $(4r + 2)$-cycle in $G \times K_2$:

$$(u_1, 1)(u_2, 2) \cdots (u_{2r+1}, 1)(u_1, 2)(u_2, 1) \cdots (u_{2r+1}, 2)(u_1, 1).$$



## *Conjectures*

# Conjecture by Y.F. Zhao[1]

**Conjecture 1**

For any simple graph $G$,

$$P(G \times K_2, k) \geq P(G \sqcup G, k) = P(G,k)^2, \quad \forall k = 1,2,3,\ldots.$$

**Conjecture 2**

For any simple graph $G$,

$$\alpha_i(G \times K_2) \geq \alpha_i(G \sqcup G), \quad \forall i = 1,2,3,\ldots.$$

Conjecture 2 $\Rightarrow$ Conjecture 1.

[1] Y.F. Zhao, the bipartite swapping trick on graph homomorphisms, *SIAM J. DISCRETE MATH.* **25** (2011), 660–680.



# Known results on the conjectures

- (Zhao) If $G$ is a graph of order $n$, then

$$P(G \times K_2, k) \geq P(G \sqcup G, k), \quad \forall k \geq (2n)^{2n+2}.$$

- If $G$ is not bipartite, the odd girth $og(G)$ is defined to be the size of a smallest odd cycle in $G$.
- (Zhao) If $G$ is a graph of order $n$, then

$$\begin{aligned} \alpha(G \sqcup G, i) &= \alpha(G \times K_2, i), \quad \textit{for } i \geq 2n - og(G) + 2; \\ \alpha(G \sqcup G, i) &< \alpha(G \times K_2, i), \quad \textit{for } i = 2n - og(G) + 1. \end{aligned}$$



# Conjecture 1 holds for chordal graphs

- A cycle $C$ in a graph $G$ is called a *chordless cycle* if $|C| \geq 4$ and $C$ is an induced subgraph of $G$.
- A graph $G$ is called a *chordal graph* if $G$ does not have any chordless cycle.
- (Dirac) A graph $G$ is chordal if and only if it has a *perfect elimination ordering* $v_1, v_2, \cdots, v_n$ of its vertices i.e., $d(v_i) = 0$ or $N(v_i) \cap \{v_1, v_2, \ldots, v_i\}$ is a clique of $G$ for all $i = 1, 2, \cdots, n$.
- (Dong) Conjecture 1 holds for chordal graphs $G$, i.e.,

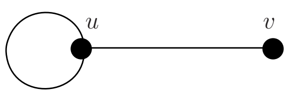
$$P(G \times K_2, k) \geq P(G \sqcup G, k), \quad \forall k = 1,2,3,\ldots.$$



*Generalization*

# Homomorphisms

- For graphs $G$ and $H$ (allowing loops for $H$), a graph homomorphism from $G$ to $H$ is a map $f : V(G) \mapsto V(H)$ such that $f$ carries each edge of $G$ to some edge of $H$, i.e., $f(u)f(v) \in E(H)$ for each $uv \in E(G)$.
- Let $Hom(G, H)$ be the set of homomorphisms from $G$ to $H$.
- For example, if $H$ is the graph with vertex set $\{u, v\}$ and edge set $\{uu, uv\}$, then for each $f \in Hom(G, H)$, $f^{-1}(v)$ is an independent set of $G$.




graph $H$



# When $Hom(G, H) = \emptyset$

- Note that for any $f \in Hom(G, H)$ and vertex $v$ in $H$ without loops at $v$, $f^{-1}(v)$ is an independent set of $G$.
- It follows that

*$H$ is simple and $Hom(G, H) \neq \emptyset$* $\Rightarrow \chi(G) \leq \chi(H)$

Or

*$H$ is simple and $\chi(G) > \chi(H)$* $\Rightarrow Hom(G, H) = \emptyset$

- Example:

*If $H$ is bipartite, then either $G$ is bipartite or $Hom(G, H) = \emptyset$.*



# $hom(G, H)$: the number of homomorphisms

- Let $hom(G, H) = |Hom(G, H)|$.
- If $H$ is the graph with vertex set $\{u, v\}$ and edge set $\{uu, uv\}$, then

  $hom(G, H) =$ the number of independent sets in $G$
- If $H = K_k$, then each member $f$ in $Hom(G, H)$ represents a proper $k$-coloring of $G$. Thus,

$$hom(G, K_k) = P(G, k).$$



# Upper bound of $hom(G, H)$ for regular and bipartite $G$

- Upper bound of $hom(G, H)$:

**Theorem (Galvin and Tetali, 2004)**

For any $d$-regular bipartite graph $G$ of order $n$,

$$hom(G, H) \leq hom(K_{d,d}, H)^{n/(2d)}.$$

- When $G$ is vertex-disjoint union of $K_{d,d}$'s, then

$$hom(G, H) = hom(K_{d,d}, H)^{n/(2d)}.$$

# GT-graphs

- A graph $H$ (not necessarily simple) is called *GT* if

$$hom(G,H) \leq hom(K_{d,d},H)^{n/(2d)}$$

holds for every $d$-regular graph $G$ of order $n$.

- Conjecture

**Conjecture (Zhao)**

For any $k \geq 3$, $K_k$ is *GT*. Equivalently, for any $d$-regular graph $G$ of order $n$:

$$P(G,k) \leq P(K_{d,d},k)^{n/(2d)}.$$

- The equality in the above conjecture holds if $G$ is the disjoint union of $K_{d,d}$'s.



# Strong GT-graphs

- A graph $H$ (not necessarily simple) is called *strong GT* if

$$hom(G \sqcup G,H) \leq hom(G \times K_2,H)$$

holds for every graph $G$ of order $n$ (not necessarily regular).

- If $H$ is strongly GT, then it is GT. It is because for any $d$-regular graph $G$ which is strong GT,

$$\begin{aligned} hom(G,H)^2 &= hom(G \sqcup G,H) \leq hom(G \times K_2,H) \\ &\leq hom(K_{d,d},H)^{n/d}. \end{aligned}$$

- Zhao guessed that there exist graphs which are GT but not strongly GT. No examples are known.



# Is $K_k$ strong GT?

**Conjecture (Zhao)**

Any complete graph $K_k$ is strong GT. Equivalently,

$$P(G \sqcup G,k) \leq P(G \times K_2,k)$$

holds for all positive integers $k$.

*Every bipartite graph is strongly GT.*

**Problems**

(1) Which graphs are GT?
(2) Which graphs are strong GT?



# Thanks for your attendance

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