An Introduction to DP color functions

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Outline

Notations

- $G = (V(G), E(G))$.
- $\mathbb{N}$: the set of positive integers.
- $\forall m \in \mathbb{N}$, let $[m] := \{1, 2, \ldots, m\}$.
- Note: in this talk, m doesn't represent the number of edges.

Proper coloring

- A **proper coloring**: a mapping $c : V(G) \to \mathbb{N}$, such that $c(u) \neq c(v)$ for all $uv \in E(G)$.
- A **proper m-coloring**: a proper coloring c with $c(u) \in [m]$ for all $u \in V(G)$.

Figure: Two proper 2-colorings of C_4

- The **chromatic polynomial** $P(G, m)$: the number of proper m-colorings, for each $m \in \mathbb{N}$.

List coloring

- Introduced by Vizing and Erdős, Rubin, Taylor independently.
- An **m-list assignment L**: a mapping L from $V(G)$ to $2^{\mathbb{N}}$, such that $|L(v)| = m$ holds for all $v \in V(G)$.
- Examples:

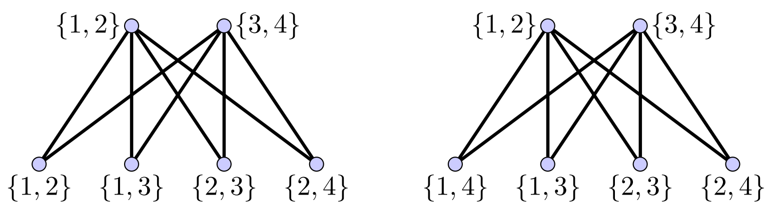


Figure: 2-list assignments of $K_{2,4}$

- $L(v) := [m]$ for all $v \in V(G)$.

List coloring

- An **L-coloring**: a proper coloring c with $c(v) \in L(v)$ for all $v \in V(G)$.
- Examples:

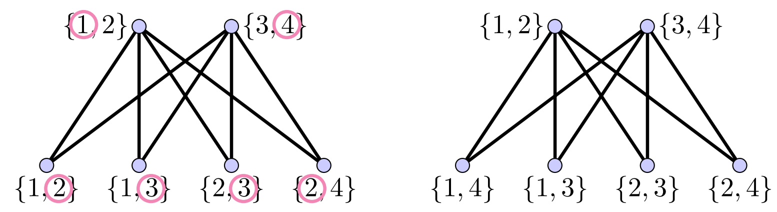


Figure: $K_{2,4}$ with a 2-list assignment L

- for the L with $L(v) = [m]$ for all $v \in V(G)$, each proper m-coloring is an L-coloring.

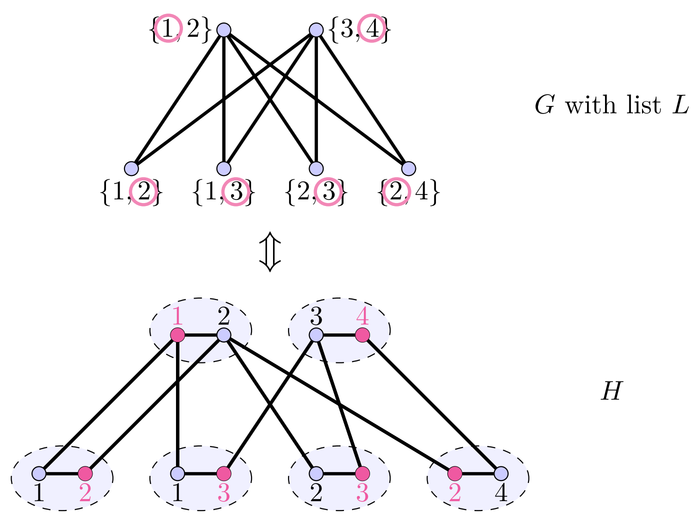
List coloring

- $P(G, L)$: the number of L-colorings.
- The **list color function** $P_l(G, m)$: the minimum value of $P(G, L)$ among all m-list assignments L, for each $m \in \mathbb{N}$.
- By definition,

$$P_l(G, m) \leq P(G, m), \ \forall\ m \in \mathbb{N}. \tag{1}$$

From list coloring to DP coloring

- Introduced by Dvořák and Postle in 2018.

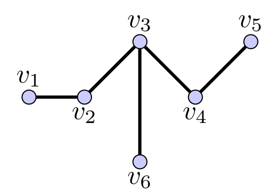


DP coloring

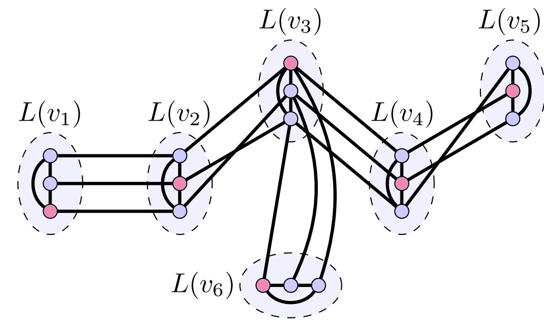
- $E_G(U,V) := \{uv \in E(G) : u \in U, v \in V\}$.
- An ***m*-fold cover**: an ordered pair $\mathcal{H} = (L, H)$, where H is a graph and L is a mapping from $V(G)$ to $2^{V(H)}$ satisfying the conditions below:
  - for every $u \in V(G)$, $L(u)$ is an m-clique in H,
  - the set $\{L(u) : u \in V(G)\}$ is a partition of $V(H)$,
  - if $uv \notin E(G)$, then $E_H(L(u), L(v)) = \emptyset$, and
  - if $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching.

DP coloring

T with a 3-fold cover $\mathcal{H} = (L, H)$.



(a) T

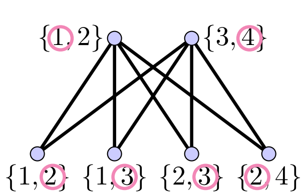


(b) H

- an **$\mathcal{H}$-coloring**: an independent set in H with size $|V(G)|$.

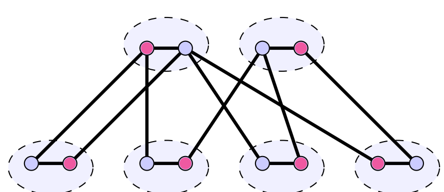
DP coloring

- For every m-list assignment L, there is a corresponding m-fold cover $\mathcal{H} = (L', H)$:
  - $V(H) = \bigcup_{u \in V(G)} L'(u)$,
  - $L'(u) = \{(u,i) : i \in L(u)\}$ for every $u \in V(G)$, and
  - $(u,i)(v,j) \in E(H)$ iff $u = v$, or $uv \in E(G)$ and $i = j$.



(a) an L-coloring

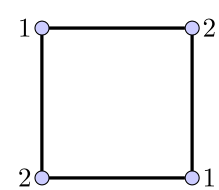
$\Leftrightarrow$



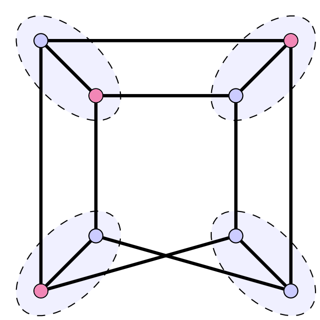
(b) an $\mathcal{H}$-coloring

DP coloring

C_4 with a 2-fold cover $\mathcal{H} = (L, H)$.



(a) C_4



(b) $\mathcal{H}$

DP coloring

- $P(G, \mathcal{H})$: the number of $\mathcal{H}$-colorings.
- The **DP color function** $P_{DP}(G, m)$: the minimum value of $P(G, \mathcal{H})$ among all m-fold covers $\mathcal{H}$, for each $m \in \mathbb{N}$.
- By definition,

$$P_{DP}(G, m) \leq P_l(G, m) \leq P(G, m), \ \forall\, m \in \mathbb{N}. \quad (2)$$

- Note that all the equalities in (2) hold when G is a chordal graph.

Three color functions

$$P_{DP}(G, m) \leq P_l(G, m) \leq P(G, m), \ \forall\, m \in \mathbb{N}.$$

Donner, 1992

$P_l(G, m) = P(G, m)$ holds when m is sufficiently large.

Thomassen, 2009

$P_l(G, m) = P(G, m)$ holds when $m > |V(G)|^{10}$.

Wang, Qian and Yan, 2017

$P_l(G, m) = P(G, m)$ holds when $m > \frac{|E(G)|}{\log(1+\sqrt{2})} \approx 1.135(|E(G)| - 1)$.

Three color functions

$$P_{DP}(G, m) \leq P_l(G, m) \leq P(G, m), \ \forall\, m \in \mathbb{N}.$$

F.M. Dong and M.Q. Zhang, 2023

$P_l(G, m) = P(G, m)$ holds whenever $m \geq |E(G)| - 1$.

- However, the DP color functions of some graphs might not tend to be the same as their chromatic polynomials.

Kaul and Mudrock, 2019

If the girth of G is even, then there exists $M \in \mathbb{N}$, such that $P_{DP}(G, m) < P(G, m)$ for all integers $m \geq M$.

Between $P_{DP}(G, m)$ and $P(G, m)$

- Therefore, two sets of graphs $\mathbf{DP}_{<}$ and $\mathbf{DP}_{\approx}$ are naturally defined.
  - $DP_{<}$: the set of graphs G for which there is $M \in \mathbb{N}$ such that $P_{DP}(G, m) < P(G, m)$ holds for all integers $m \geq M$, and
  - $DP_{\approx}$: the set of graphs G for which there is $M \in \mathbb{N}$ such that $P_{DP}(G, m) = P(G, m)$ holds for all integers $m \geq M$.
- Note: a characterization of the graphs in set $DP_{<}$ or $DP_{\approx}$ does not necessarily guarantee a characterization of the graphs in the other set.

Known results on $DP_<$

- For any $e \in E(G)$, let $\mathcal{C}_G(e)$ be the set of shortest cycles in G containing e.
- The **girth of edge** e, denoted by $\ell_G(e)$:
  - ∞ if $\mathcal{C}_G(e) = \emptyset$;
  - the size of any cycle in $\mathcal{C}_G(e)$ otherwise.

Dong and Yang, 2022

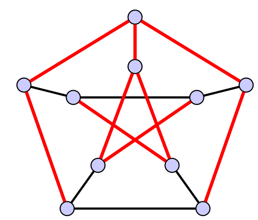
Graph G belongs to $DP_<$ if G contains an edge whose girth is even.

Known results on $DP_\approx$

- On the other hand, Mudrock and Thomason first showed that each graph with a dominating vertex belongs to $DP_\approx$.

Dong and Yang, 2022

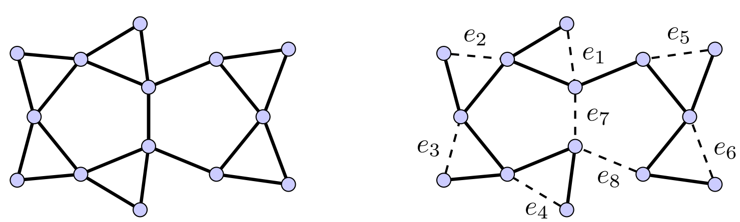
Graph G belongs to $DP_\approx$ if G has a spanning tree T such that for each edge e in $E(G) \setminus E(T)$, $\ell_G(e)$ is odd and there exists a cycle $C \in \mathcal{C}_G(e)$, where $\ell_G(e') < \ell_G(e)$ holds for each $e' \in E(C) \setminus (E(T) \cup \{e\})$.



Our results on $DP_\approx$

M.Q. Zhang and F.M. Dong, 2023

Graph G belongs to $DP_\approx$ if G has a spanning tree T and a labeling $e_1, \ldots, e_q$ of all the edges in $E(G) \setminus E(T)$, such that $\ell_G(e_1) \leq \cdots \leq \ell_G(e_q)$ and for each $i \in [q]$, $\ell_G(e_i)$ is odd and $E(C_{e_i}) \subseteq E(T) \cup \{e_1, \ldots, e_i\}$ holds for some $C_{e_i} \in \mathcal{C}_G(e_i)$.



(a) G (b) T and an edge labeling

Our results on $DP_\approx$

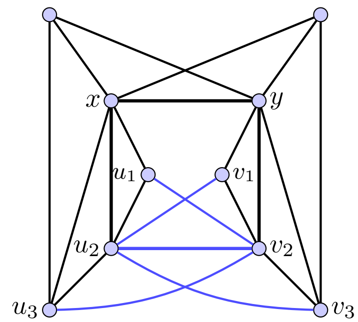
M.Q. Zhang and F.M. Dong, 2023

Let G be a graph with vertex set $\{v_i : i = 0, 1, \ldots, n\}$, where $n \geq 1$. If for each $i \in [n]$, the set $N(v_i) \cap \{v_j : 0 \leq j \leq i-1\}$ is not empty and the subgraph of G induced by this vertex set is connected, then G is in $DP_\approx$.

- Immediately, many special classes of graphs belong to $DP_\approx$, such as complete k-partite graphs with $k \geq 3$ and plane near-triangulations.

Our results on $DP_<$

- For any $E^* \subseteq E(G)$, let $\mathcal{C}_G(E^*)$ be the set of shortest cycles C in G such that $|E(C) \cap E^*|$ is odd.
- The **girth of edge set** E^*, denoted by $\ell_G(E^*)$:
  - ∞ if $\mathcal{C}_G(E^*) = \emptyset$;
  - the size of any cycle in $\mathcal{C}_G(E^*)$ otherwise.

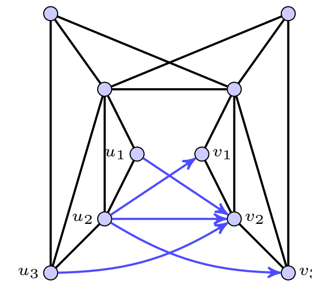


$E^* = \{u_1v_2, u_2v_1, u_2v_2, u_2v_3, u_3v_2\}$

$\ell_G(E^*) = 4$

Our results on $DP_<$

- For any $E^* \subseteq E(G)$, assume that each e in E^* is assigned a direction $\overrightarrow{e}$ and only edges in E^* are assigned directions.

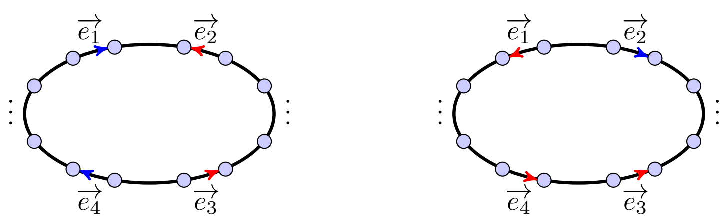


- Let $\overrightarrow{E^*}$ be the set of directed edges $\overrightarrow{e}$ for all $e \in E^*$.

$\Rightarrow \overrightarrow{E^*} = \{\overrightarrow{u_1v_2}, \overrightarrow{u_2v_1}, \overrightarrow{u_2v_2}, \overrightarrow{u_2v_3}, \overrightarrow{u_3v_2}\}.$

Our results on $DP_<$

- For any cycle C, we say the directed edges in $\overrightarrow{E^*}$ are **balanced on** C if $|E(C) \cap E^*|$ is even, and exactly half of the edges in $E(C) \cap E^*$ are oriented clockwise along C.



(a) Balanced (b) Unbalanced

Figure: $\overrightarrow{E^*} = \{\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}, \overrightarrow{e_4}\}$

Our results on $DP_<$

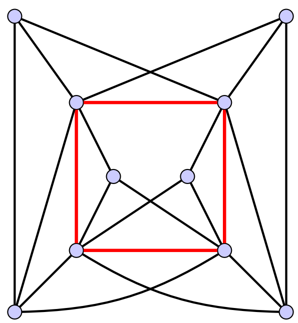
M.Q. Zhang and F.M. Dong, 2023

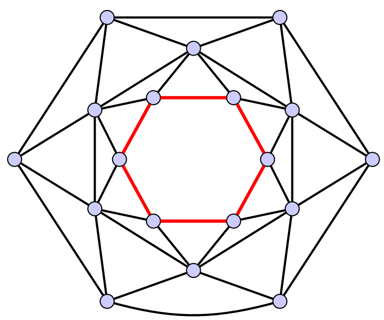
Let G be a connected graph and E^* be a set of edges in G. Assume that

1. $\ell_G(E^*)$ is even; and
2. there exists a way to assign a direction $\overrightarrow{e}$ for each edge $e \in E^*$ such that the directed edges in $\overrightarrow{E^*} = \{\overrightarrow{e} : e \in E^*\}$ are balanced on each cycle C of G with $|E(C)| < \ell_G(E^*)$.

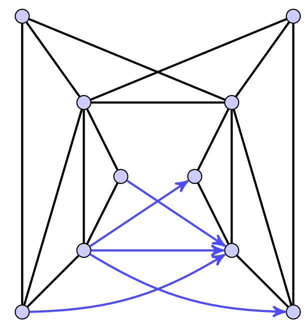
Then $P(G, m) - P_{DP}(G, m) \geq \Omega(m^{|V(G)| - \ell_G(E^*) + 1})$ holds, and hence $G \in DP_<$.

Our results on $DP_<$

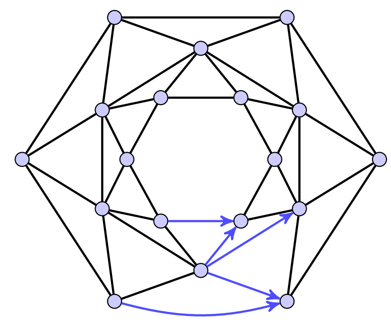




Our results on $DP_<$



$\ell_G(E^*) = 4$



$\ell_G(E^*) = 6$

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Let G be any graph and let $E^* \subseteq E_G(V_1, V_2)$, where V_1 and V_2 are disjoint vertex subsets of $V(G)$. If $\ell_G(E^*) = 4$, then $P(G,m) - P_{DP}(G,m) \geq \Omega(m^{n-3})$ holds, and hence $G \in DP_<$.

Future Research

Question

How to characterize the graphs in sets $DP_<$ and $DP_\approx$?

Question

What is the property of an m-fold cover $\mathcal{H}$ of G with $P(G,\mathcal{H}) = P_{DP}(G,m)$?

Question

Is there any graph not in $DP_< \cup DP_\approx$?

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