Study on tiered trees

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Outline

Article

The main results in this talk is from the following article:

Fengming Dong and Sherry H.F. Yan, Proving identities on weight polynomials of tiered trees via Tutte polynomials,
J. Combin. Theory Ser. A **193** *(2023), 105689.*
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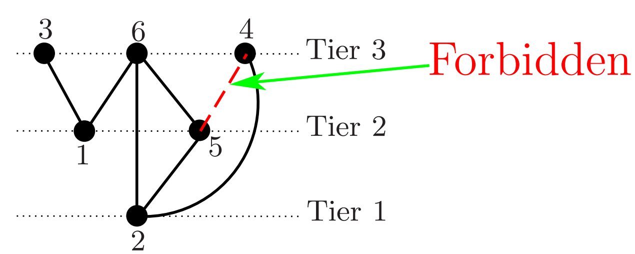
Tiered graphs

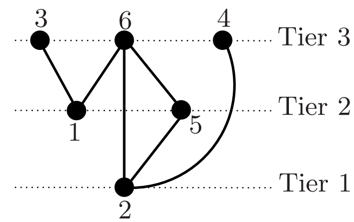


Rules for tiered graphs:
(a) Vertices are denoted by positive integers and located in tiers;
(b) Vertices in the same tier form an independent set; and
(c) If $uv \in E$ and $u > v$, then u is in a higher tier than v.

A resident building

Each tiered graph has a tiering map t

A *tiered graph* $G = (V, E)$ with $m \geq 2$ tiers is a simple graph with $V \subseteq [\![n]\!] = \{1, 2, \cdots, n\}$, and with a surjective map t from V to $[\![m]\!]$ such that if $vv' \in E$, then $v > v'$ implies $t(v) > t(v')$.



For this example

i	1	2	3	4	5	6
$t(i)$	2	1	3	3	2	3

t is called a *tiering map*, which decides the tier in which each vertex i in G is located.

Tiered trees

- If a tiered graph is a tree, it is called a *tiered tree*.
- The concept of tiered trees was introduced by Dugan et al. in 2019 who generalized the concept alternating trees introduced by Postnikov in 1997.
- A tree is called *an alternating tree* if for each path $x_1 x_2 \ldots x_k$, either

$$x_1 < x_2 > x_3 < x_4 > x_5 \cdots$$

or

$$x_1 > x_2 < x_3 > x_4 < x_5 \cdots$$

- Any path in a tiered graph with 2 tiers is an alternating path.

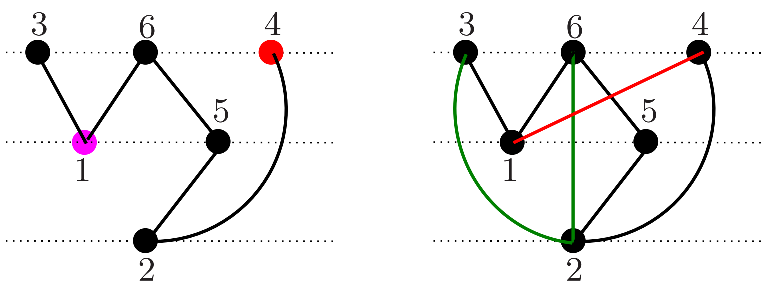
Connection to other combinatorial objects

- the regions of the Linial hyperplane arrangement (the affine arrangement in $\mathcal{R}^n$ defined by the equations $x_i - x_j = 1, 1 \leq i < j \leq n$);
- local binary search trees (labeled plane binary trees with the property that every left child has a smaller label than its parent and every right child has a larger label than its parent);
- semiacyclic tournaments (directed graphs on the set $\{1, \ldots, n\}$ such that in every directed cycle, there are more edges (i, j) with $i < j$ than with $i > j$).

Complete tiered graphs

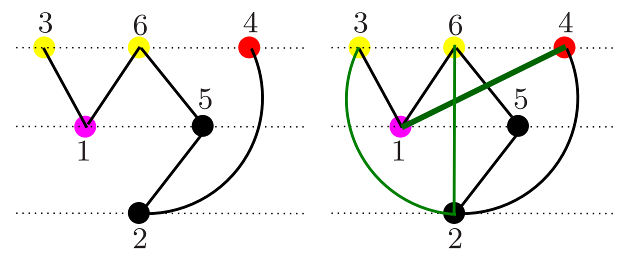
A tiered graph G with a tiering map t is *complete* if

$$u < v,\ t(u) < t(v) \Rightarrow uv \in E(G).$$



(a) not complete (b) complete

Complete tiered graphs



(a) Tiered tree T (b) Tiered graph $G = K_T$

Note that the tiered graph G is determined by T, denoted by K_T.

K_T is the completed tiered graph obtained from T by adding some new edges.

Counting problems

Counting Tiered trees

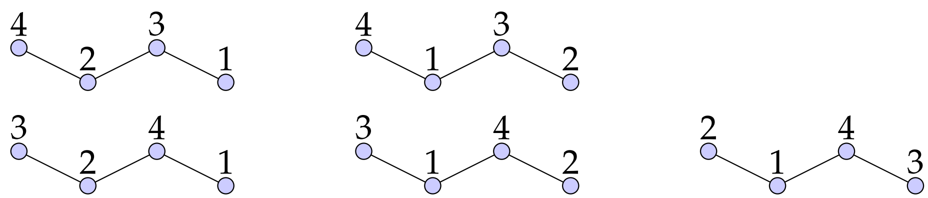
- Two tiered graphs G_1 and G_2 are *different* if either $E(G_1) \neq E(G_2)$ or G_1 and G_2 have different tiering functions.
- Let $\mathcal{T}_{n,m}$ be the set of tiered trees with n vertices and m tiers.
- Postnikov (1997): $|\mathcal{T}_{n,2}| = \frac{1}{n2^{n-1}} \sum_{k\geq 1} \binom{n}{k} k^{n-1}$.
- Dugan et al. (2019):

$$|\mathcal{T}_{n,m}| = \frac{1}{nm^{n-1}} \sum_{\substack{k_i \geq 0 \\ k_1+\cdots+k_m=n}} \binom{n}{k_1,\ldots,k_m} \left(\sum_{i=1}^{m} (m-i)k_i\right)^{n-1}.$$

The set $\mathcal{T}_{\mathbf{p}}$ of tiered trees

For any ordered partition $\mathbf{p} = (p_1, \cdots, p_m)$ of n, let $\mathcal{T}_{\mathbf{p}}$ be the set of tiered trees with exactly p_i vertices in tier i.

Exactly 5 tiered trees in $\mathcal{T}_{(2,2)}$:



(a) (b) (c)

Open problems

Problem: *Given a partition $\mathbf{p} = (p_1, p_2, \cdots, p_m)$ of n, find an expression for $|\mathcal{T}_{\mathbf{p}}|$ in terms of $p_1, p_2, \cdots, p_m$.*

Special case:

Problem: *Given any partition $\mathbf{p} = (p_1, p_2)$ of n, determine $|\mathcal{T}_{\mathbf{p}}|$ in terms of p_1 and p_2.*

$\mathcal{T}_{\mathbf{p}}$ and $\mathcal{T}_{\pi(\mathbf{p})}$ have the same size

- Sherry H.F. Yan, Danna Yan, Hao Zhou (DM, 2020):

For any ordered partition $\mathbf{p} = (p_1, p_2, \cdots, p_m)$ of n, $|\mathcal{T}_{\mathbf{p}}| = |\mathcal{T}_{\pi(\mathbf{p})}|$ holds for any permutation of π of $1, 2, \cdots, m$, where $\pi(\mathbf{p}) = (p_{\pi(1)}, p_{\pi(2)}, \cdots, p_{\pi(m)})$.

- For example, $|\mathcal{T}_{(1,2,3)}| = |\mathcal{T}_{(3,2,1)}|$.
- In other words, for any partition $\mathbf{p} = (p_1, p_2, \cdots, p_m)$ of n, $|\mathcal{T}_{\mathbf{p}}|$ is independent of the order of $p_1, p_2, \cdots, p_m$.

Weight $w(T)$ of a tiered tree T

More details than $|\mathcal{T}_{\mathbf{p}}| = |\mathcal{T}_{\pi(\mathbf{p})}|$

$w(T)$: the weight of a tiered tree T.

We shall prove that

For any partition $\mathbf{p} \vdash n$, permutation π, and $i = 0, 1, 2, \ldots,$

$$|\{T \in \mathcal{T}_{\mathbf{p}} : w(T) = i\}| = |\{T \in \mathcal{T}_{\pi(\mathbf{p})} : w(T) = i\}|.$$

Equivalently, it is to prove the following identity:

$$\sum_{T \in \mathcal{T}_{\mathbf{p}}} y^{w(T)} = \sum_{T \in \mathcal{T}_{\pi(\mathbf{p})}} y^{w(T)}.$$

Example

Assume that

$\mathcal{T}_{(a,b,c)}$ has 40 tiered trees

$w(T)$	0	1	2	≥ 3
No. T	10	12	18	0

$\mathcal{T}_{(b,c,a)}$ has 40 tiered trees

$w(T)$	0	1	2	≥ 3
No. T	10	12	18	0

Equivalently,

$$\sum_{T \in \mathcal{T}_{\mathbf{p}}} y^{w(T)} = 10 + 12y + 18y^2 = \sum_{T \in \mathcal{T}_{\pi(\mathbf{p})}} y^{w(T)}$$

How is the weight $w(T)$ defined?

Externally active edges

G: a connected graph with a weight function μ on $E(G)$, which is a real and injective function.

μ provides an order for edges in G.

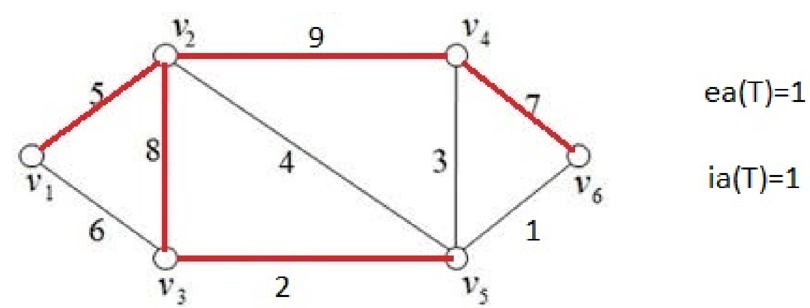
T is a spanning tree of G.

For any edge $e \in E(G) \setminus E(T)$, $T + e$ has a unique cycle, denoted by $C_T(e)$, with respect to T.

For $e \in E(G)$, e is said to be *externally active* in G with respect to T, if $e \notin E(T)$ and

$$\mu(e) \leq \mu(e'), \quad \forall e' \in E(C_T(e)).$$

The external activity $ea_G(T)$ for a spanning tree T

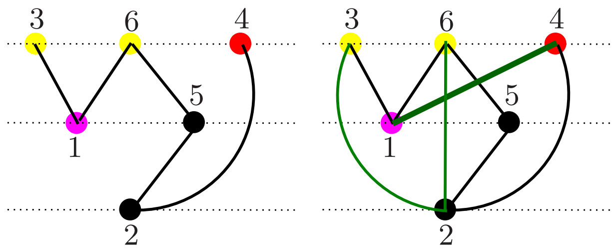


External activity of T in G, denoted by $ea_G(T)$: the number of externally active edges in G with respect to T.

$ea(T) = 1$ for the above example, since edge v_5v_6 with weight 1 is the only externally active edge in G with respect to T.

Weight function μ for tiered graphs

For each tiered graph G, the edges in G are ordered lexicographically by their endpoints.



(a) Tiered tree T (b) Tiered graph $G = K_T$

Thus, for the above tiered graph G,

$$\mu(1,3) < \mu(1,4) < \mu(1,6) < \mu(2,3) < \mu(2,5) < \mu(2,6) < \mu(5,6).$$

The weight $w(T)$ of a tiered tree T

Let T be a tiered tree with tiering map t. The weight $w(T)$ is the number of $i - j$ paths P in T such that

(1) $i < v$ for each $v \in V(P) \setminus \{i\}$;

(2) $|V(P)| \geq 3$;

(3) $t(j) > t(i)$; and

(4) $k > j$, where k is the neighbor of i on path P.

Equivalent conditions

(1) $i < v$ for each $v \in V(P) \setminus \{i\}$;
(2) $|V(P)| \geq 3$;
(3) $t(j) > t(i)$; and
(4) $k > j$, where k is the neighbor of i on path P.

The following statements are equivalent:

(a) path P satisfies conditions (1)-(4) above;

(b) $\mu(i,j) \leq \mu(u,v)$ for each edge (u,v) on P;

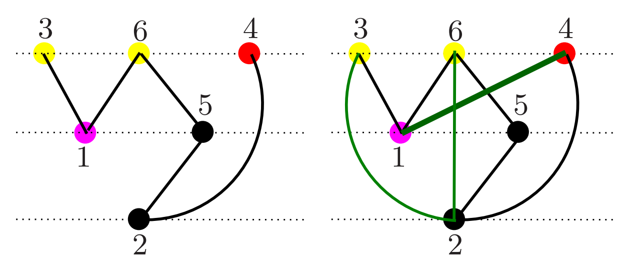
(c) edge (i,j) is externally active in K_T with respect T.

$w(T) = ea_{K_T}(T)$

For a tiered tree T, $w(T)$ is equal to the external activity of T in K_T:

$$w(T) = ea_{K_T}(T).$$

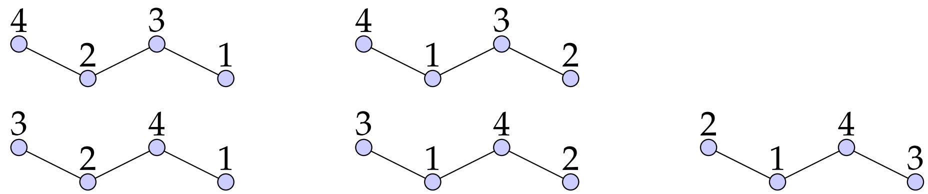
For the following tiered tree T, $w(T) = ea_{K_T}(T) = 1$.



(a) T (b) K_T

Weight $w(T)$ for $T \in \mathcal{T}_{(2,2)}$

Determine the weight $w(T)$ for each tree T in $\mathcal{T}_{(2,2)}$:



(a) (b) (c)

$w(T) = 1$ for only one tree T above, $w(T) = 0$ for all other trees.

$w(T) = 1$ for which tree T?

$w(T) = 1$ for the tree T on the bottom of (a).

Problem on the weight polynomial $P_{\mathbf{p}}(y)$

The weight polynomial $P_{\mathbf{p}}(y)$

- For any ordered partition $\mathbf{p} = (p_1, \cdots, p_m)$ of n, the weight polynomial for trees in $\mathcal{T}_{\mathbf{p}}$ is defined as

$$P_{\mathbf{p}}(y) = \sum_{T \in \mathcal{T}_{\mathbf{p}}} y^{w(T)}.$$

- Since $w(T) = ea_{K_T}(T)$,

$$P_{\mathbf{p}}(y) = \sum_{T \in \mathcal{T}_{\mathbf{p}}} y^{ea_{K_T}(T)}.$$

- For example, if $\mathbf{p} = (2,2)$, then $P_{\mathbf{p}}(y) = y + 4$.

Problem asked by Dugan et al

Problem

Is there an elementary proof of the identity below for any partition $\mathbf{p} = (p_1, \cdots, p_m)$ and any permutation π of $1, 2, \cdots, m$,

$$P_{\mathbf{p}}(y) = P_{\pi(\mathbf{p})}(y)$$

i.e.,

$$\sum_{T \in \mathcal{T}_{\mathbf{p}}} y^{w(T)} = \sum_{T \in \mathcal{T}_{\pi(\mathbf{p})}} y^{w(T)}?$$

For example, proving the following identity:

$$P_{(1,2,3,4,5)}(y) = P_{(5,4,3,2,1)}(y).$$

Transferred to Tutte polynomial

- For any ordered partition $\mathbf{p} = (p_1, \cdots, p_m)$ of n.
- Let $\mathcal{CG}_{\mathbf{p}}$ be the set of completed tiered graphs G with tiering map t such that $t^{-1}(i) = p_i$ for $i = 1, \cdots, m$.
- Let $\mathcal{CG}^c_{\mathbf{p}}$ be the set of connected graphs in $\mathcal{CG}_{\mathbf{p}}$.
- The weight polynomial can be transferred to

$$P_{\mathbf{p}}(y) = \sum_{T \in \mathcal{T}_{\mathbf{p}}} y^{w(T)} = \sum_{G \in \mathcal{CG}^c_{\mathbf{p}}} \mathbf{T}_G(1, y),$$

where $\mathbf{T}_G(x, y)$ is the Tutte polynomial of G:

$$\mathbf{T}_G(x, y) = \sum_{T \in \mathcal{ST}_G} x^{ia(T)} y^{ea(T)}.$$

Tutte polynomial $\mathbf{T}_G(x, y)$

It is named after William Tutte (1917-2002).

Let $G = (V, E)$ be a undirected graph.

For any $A \subseteq E$, let $k(A)$ denote the number of components of the spanning subgraph (V, A).

The Tutte polynomial of G is defined as

$$\begin{aligned} \mathbf{T}_G(x, y) &:= \sum_{A \subseteq E} (x-1)^{k(A)-k(E)} (y-1)^{k(A)+|A|-|E|} \\ &= \sum_{T \in \mathcal{ST}_G} x^{ia(T)} y^{ea(T)}, \end{aligned}$$

where $ia(T)$ is the internal activity of T.

Equivalent problem on Tutte polynomial

Transfer of the problem

For any partition $\mathbf{p} = (p_1, \cdots, p_m)$ and any permutation π of $1, 2, \cdots, m$,

$$P_{\mathbf{p}}(x) = P_{\pi(\mathbf{p})}(x)$$

$$\Updownarrow$$

For any partition $\mathbf{p} = (p_1, \cdots, p_m)$ and any permutation π of $1, 2, \cdots, m$,

$$\sum_{G \in \mathcal{CG}^c_{\mathbf{p}}} \mathbf{T}_G(1, y) = \sum_{G \in \mathcal{CG}^c_{\pi(\mathbf{p})}} \mathbf{T}_G(1, y),$$

where $\mathcal{CG}^c_{\mathbf{p}}$ is the set of connected complete tiered graphs with tier partition $\mathbf{p}$.

Approach

- Sufficient to prove

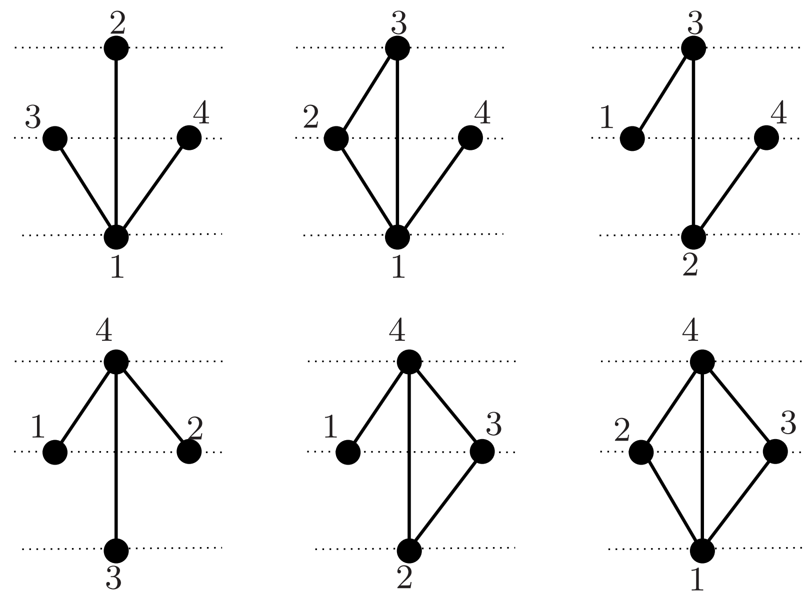
for any permutation π of $1, 2, \cdots, m$ which exchanges i and $i+1$ only, where $1 \le i < m$:

$$\sum_{G \in \mathcal{CG}^c_{\mathbf{p}}} \mathbf{T}_G(1, y) = \sum_{G \in \mathcal{CG}^c_{\pi(\mathbf{p})}} \mathbf{T}_G(1, y).$$

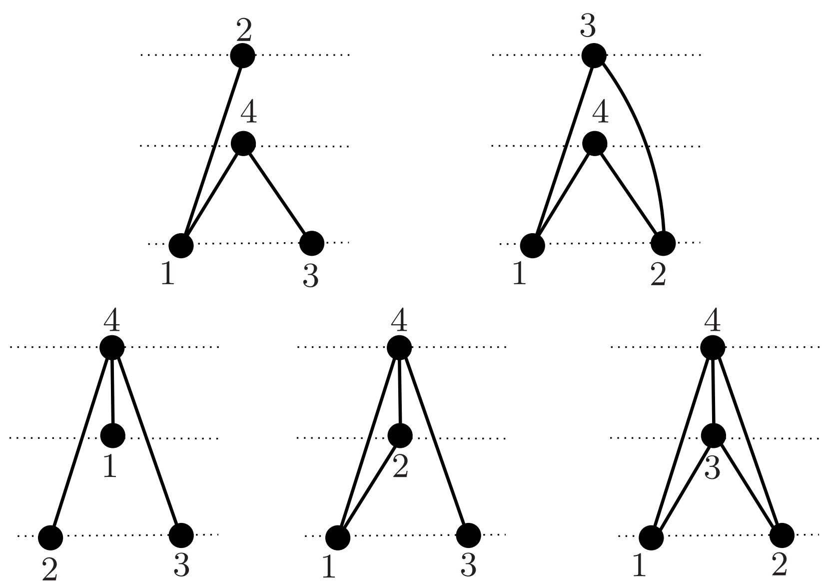
- The total number of spanning trees of graphs in the set $\mathcal{CG}^c_{\mathbf{p}}$ is equal to the number of total number of spanning trees of graphs in $\mathcal{CG}^c_{\pi(\mathbf{p})}$.

But $\left|\mathcal{CG}^c_{\mathbf{p}}\right| \neq \left|\mathcal{CG}^c_{\pi(\mathbf{p})}\right|$ for some $\mathbf{p}$, e.g., $\left|\mathcal{CG}^c_{(1,2,1)}\right| > \left|\mathcal{CG}^c_{(2,1,1)}\right|$.

Six graphs in the set $\mathcal{CG}^c_{(1,2,1)}$



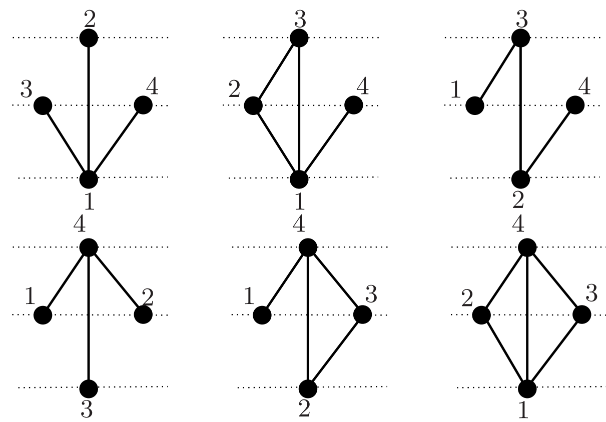
Five graphs in the set $\mathcal{CG}^c_{(2,1,1)}$



$\mathbf{T}_G(1, y)$ for $G \in \mathcal{CG}^c_{(1,2,1)}$

Sum $= y^2 + 5y + 11$

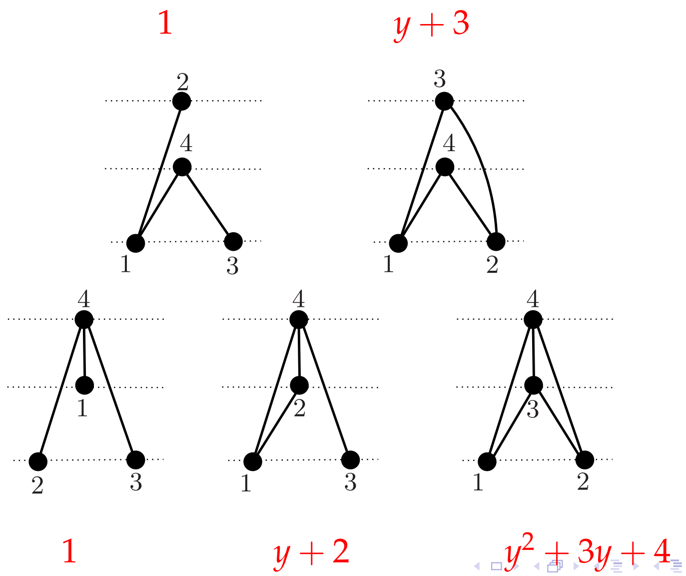
1 $\quad y+2 \quad$ 1



1 $\quad y+2 \quad$ $y^2 + 3y + 4$

$T_G(1,y)$ for $G \in \mathcal{CG}^c_{(2,1,1)}$

Sum $= y^2 + 5y + 11$



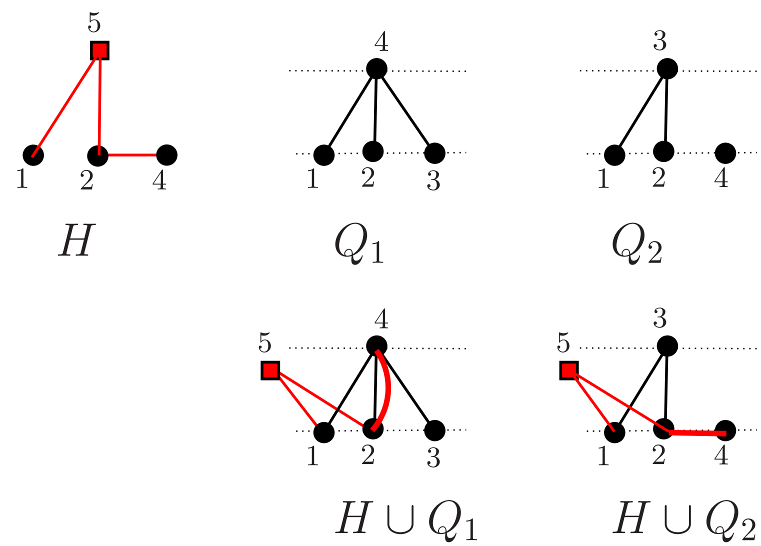
Approach of solving the problem

Union of graphs: $H \cup Q$

- Let H be a multiple graph and Q be a tiered graph, with the possibility that $V(H) \cap V(Q) \neq \emptyset$.
- $H \cup Q$ is defined to be the multi-graph with vertex set $V(H) \cup V(Q)$ and edge set $E(H) \cup E(Q)$,

  where any edges $e_1 \in E(H)$ and $e_2 \in E(Q)$ are two different edges in $H \cup Q$ even if e_1 and e_2 join the same pair of vertices.
- Thus, $|E(H \cup Q)| = |E(H)| + |E(Q)|$.

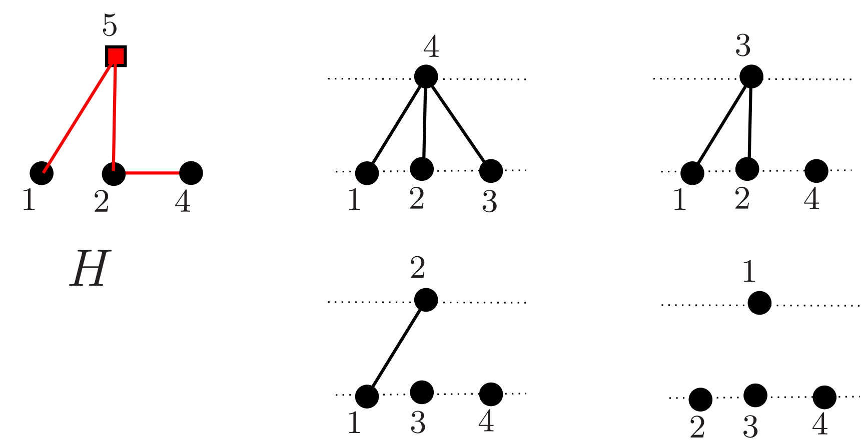
Examples of graphs $H \cup Q$



The set of graphs in $\mathcal{CG}^c_{U,\mathbf{p}}(H)$

- Let U be any subset of $[\![n]\!]$, $\mathbf{p} = (p_1, p_2)$ and $\mathbf{p}' = (p_2, p_1)$, where $p_1 + p_2 = |U|$.
- Let $\mathcal{CG}_{U,\mathbf{p}}$ be the set of complete tiered graphs Q with tiering map $t : U \to \{1, 2\}$ such that $t^{-1}(i) = p_i$ for $i = 1, 2$.
- Given any graph H, let $\mathcal{CG}^c_{U,\mathbf{p}}(H)$ be the set of connected graphs $H \cup Q$, where $Q \in \mathcal{CG}_{U,\mathbf{p}}$.

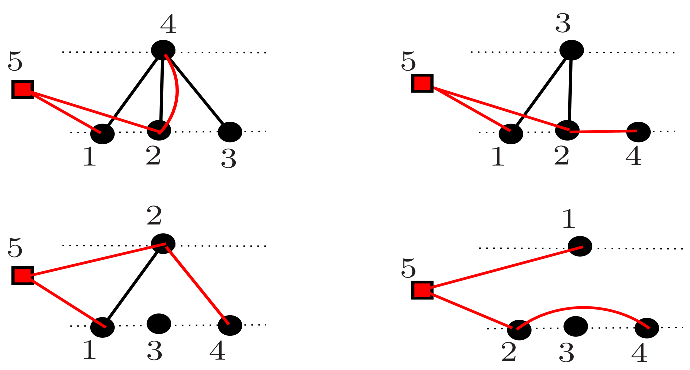
H and graphs in $\mathcal{CG}_{\{1,2,3,4\},(3,1)}$



Fours graphs in $\mathcal{CG}_{\{1,2,3,4\},(3,1)}$

Only two graphs in $\mathcal{CG}^c_{\{1,2,3,4\},(3,1)}(H)$

$\mathbf{T}_G(1, y) = y^2 + 3y + 3$ $\qquad$ $\mathbf{T}_G(1, y) = y + 3$

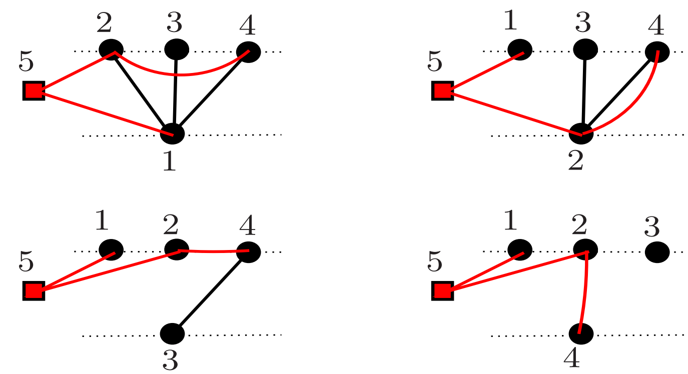


disconnected $\qquad$ disconnected

$$\sum_{G \in \mathcal{CG}^c_{\{1,2,3,4\},(3,1)}(H)} \mathbf{T}_G(1, y) = y^2 + 4y + 6.$$

Three graphs in $\mathcal{CG}^c_{\{1,2,3,4\},(1,3)}(H)$

$\mathbf{T}_G(1, y) = y^2 + 3y + 4$ $\qquad$ $\mathbf{T}_G(1, y) = y + 1$



disconnected

$$\sum_{G \in \mathcal{CG}^c_{\{1,2,3,4\},(1,3)}(H)} \mathbf{T}_G(1, y) = y^2 + 4y + 6.$$

An extension

$U \subseteq [\![n]\!]$, $\mathbf{p} = (p_1, p_2)$ and $\mathbf{p}' = (p_2, p_1)$, where $p_1 + p_2 = |U|$.

Dong and Yan (2022):
For any multi-graph H,

$$\sum_{G \in \mathcal{CG}^c_{U,\mathbf{p}}(H)} \mathbf{T}_G(1,y) = \sum_{G \in \mathcal{CG}^c_{U,\mathbf{p}'}(H)} \mathbf{T}_G(1,y).$$

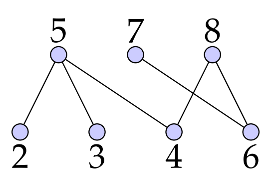
It implies that

For any ordered partition $\mathbf{p} = (p_1, \cdots, p_m)$ and any permutation π of $1, 2, \cdots, m$ exchanging i and j only, where $1 \le i < j \le m$:

$$\sum_{G \in \mathcal{CG}^c_{\mathbf{p}}} \mathbf{T}_G(1,y) = \sum_{G \in \mathcal{CG}^c_{\pi(\mathbf{p})}} \mathbf{T}_G(1,y).$$

Main idea

The dual graph of a 2-tier graph



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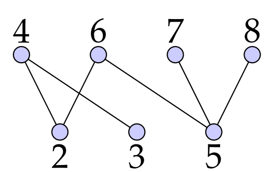


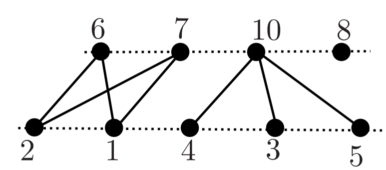
its dual graph T'

G is a connected tiered graph with vertices $x_1, x_2, \cdots, x_s$, where $x_1 < x_2 < \cdots < x_s$, and a tiering map $t : V(G) \to \{1, 2\}$.

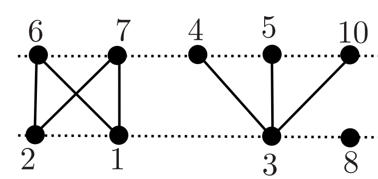
The *dual graph of* G, denoted by G', has vertex set $V(G)$, tiering map $t' : V(G') \to \{1, 2\}$ with $t'(x_r) = 3 - t(x_{s+1-r})$ for all $r = 1, 2, \cdots, s$, and edge set $\{x_i x_j : x_{s+1-i} x_{s+1-j} \in E(G), 1 \le i < j \le s\}$.

The dual graph of a tiered graph

If G is a tiered graph with components $G_1, G_2, \ldots, G_k$, then *the dual graph* of G is defined to be the tiered graph with components $G'_1, G'_2, \cdots, G'_k$.



G



G'

For a 2-tier graph G, $G' \cong G$ and $V(G_i) = V(G'_i)$ for each component G_i of G,
but it is not true that G is complete if and only if G' is.

Correspondence of spanning trees

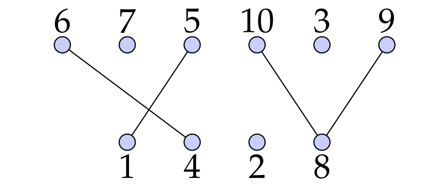
For any graph W and 2-tier forest F, $W \cup F$ is a tree if and only if $W \cup F'$ is a tree.

Thus,

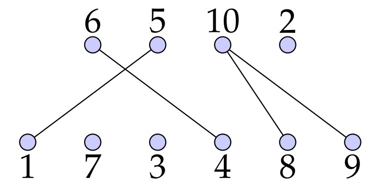
$W \cup F \Rightarrow W \cup F'$ is a bijection from

$$\bigcup_{G \in \mathcal{CG}^c_{U,(p_1,p_2)}(H)} \mathcal{ST}_G \quad \text{to} \quad \bigcup_{G \in \mathcal{CG}^c_{U,(p_2,p_1)}(H)} \mathcal{ST}_G$$

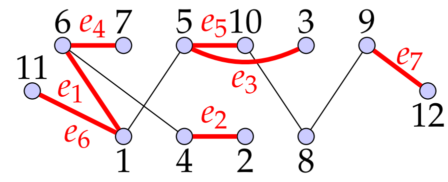
Example



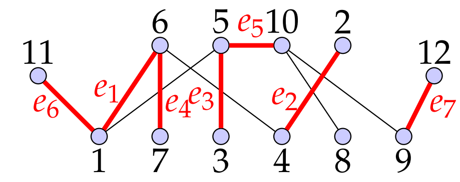
F



its dual F'



$W \cup F$



$W \cup F'$

Different weight functions μ_1 and μ_2

Next target: To find weight functions μ_1 on $E(H) \cup \bigcup_{G \in \mathcal{CG}_{U,(p_1,p_2)}} E(G)$ and μ_2 on $E(H) \cup \bigcup_{G \in \mathcal{CG}_{U,(p_2,p_1)}} E(G)$

such that for any tree $W \cup F$, where $W \subseteq H$, $F \subseteq G$, and $G \in \mathcal{CG}_{U,(p_2,p_1)}$,

the external activity of $W \cup F$ in $H \cup G$ with respect to μ_1

=

the external activity of $W \cup F'$ in $H \cup K_{F'}$ with respect to μ_2.

Note that $K_{F'}$ may be different from the dual G' of G.

Conclusion after confirming μ_1 and μ_2

Note that

$$\sum_{G \in \mathcal{CG}^c_{U,(p_1,p_2)}(H)} \mathbf{T}_G(1,y) = \sum_{G \in \mathcal{CG}^c_{U,(p_1,p_2)}(H)} \sum_{T \in \mathcal{ST}_G} y^{ea_{\mu_1}(T)}$$

and

$$\sum_{G \in \mathcal{CG}^c_{U,(p_2,p_1)}(H)} \mathbf{T}_G(1,y) = \sum_{G \in \mathcal{CG}^c_{U,(p_2,p_1)}(H)} \sum_{T \in \mathcal{ST}_G} y^{ea_{\mu_2}(T)}.$$

As μ_1 and μ_2 have the above property,

$$\sum_{G \in \mathcal{CG}^c_{U,(p_1,p_2)}(H)} \mathbf{T}_G(1,y) = \sum_{G \in \mathcal{CG}^c_{U,(p_2,p_1)}(H)} \mathbf{T}_G(1,y)$$

The main result then follows.

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Thanks for your attendance

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