List-coloring functions versus chromatic polynomials

Dong Fengming

Nanyang Technological University
Singapore

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Articles

Fengming Dong and Meiqiao Zhang, An improved lower bound of $P(G,L)-P(G,k)$ for k-assignments *L. J. Combin. Theory Ser. B* **161** (2023), 109-119. http://doi.org/10.1016/j.jctb.2023.02.002

Fengming Dong and Meiqiao Zhang, Compare the list-color function of a hypergraph with its chromatic polynomial. http://arxiv.org/abs/2212.02045

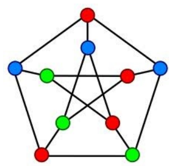
Meiqiao Zhang and Fengming Dong, Compare list-color functions of hypergraphs with their chromatic polynomials (II). http://arxiv.org/abs/2302.05067

Outline

Chromatic polynomials

Proper coloring

- For a positive integer k, a (proper) k-coloring of a graph G is a way of assigning k colors to vertices in G, one color for each vertex, such that any two adjacent vertices are assigned different colours.



Chromatic number $\chi(G)$ is the minimum k such that G admits a proper k-coloring.

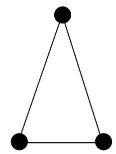
Brooks Theorem

For any connected graph G, if G is not complete nor an odd cycle, then $\chi(G) \leq \Delta(G)$.

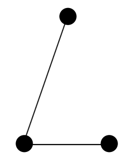
$P(G,k)$: Counting k-colourings

$P(G,k)$: the number of ways of assigning one color in $\{c_1, \cdots, c_k\}$ to each vertex of G such that any two adjacent vertices are colored differently.

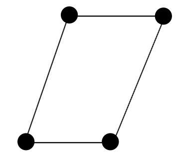
- Examples for $P(G,k)$:



$k(k-1)(k-2)$



$k(k-1)^2$



$(k-1)^4 + (k-1)$

- $P(G,k)$ is a polynomial in k of degree $|V(G)|$.
- $P(K_n,k) = k(k-1)\cdots(k-n+1)$ and $P(N_n,k) = k^n$.

Chromatic polynomial

$P(G,k)$ is called the chromatic polynomial of G.

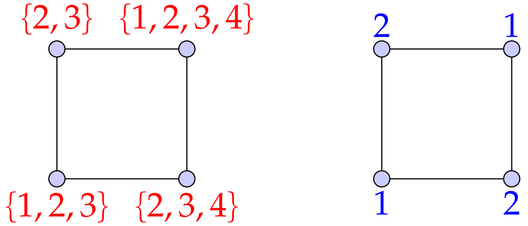
It was introduced by Birkhoff in 1912 with the hope of proving 4CC.

George David Birkhoff (1884-1944) was one of the most important leaders in American mathematics in his generation.

List-coloring function $P_l(G,k)$

List-coloring: a generalization of vertex-coloring

- Introduced independently by Vizing in 1976 and Erdős, Rubin and Taylor in 1979.
- *List assignment* of G: a mapping L from $V(G)$ to $2^{\mathbb{N}}$.
- For a list assignment L, a *proper* L-coloring of G is a mapping $f : V(G) \to \mathbb{N}$ such that $f(v) \in L(v)$ for each $v \in V(G)$ and $f(u) \neq f(v)$ for each edge $uv \in E(G)$.



$P(G, L)$: the number of L-colorings

- Given a list assignment L, let $P(G, L)$ be the number of proper L-colorings.
- If $L(v) = \{1, 2, \cdots, k\}$ for each vertex v in G, then $P(G, L) = P(G, k)$.
- In fact, if $L(u) = L(v)$ for every edge uv in G, then $P(G, L) = P(G, k)$.

Computation of $P(G, L)$

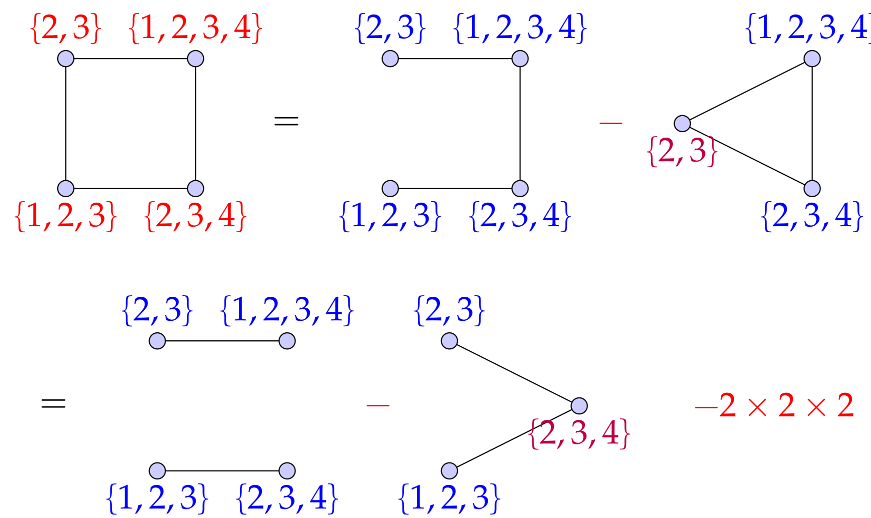
- If G is an empty graph, then
$$P(G, L) = \prod_{v \in V(G)} |L(v)|.$$
- If G is disconnected with components $G_1, G_2, \cdots, G_c$, then
$$P(G, L) = \prod_{v \in V(G)} P(G_i, L).$$

Deletion-contraction Theorem: *For any edge $e = v_1v_2$ in G,*

$$P(G, L) = P(G \backslash e, L) - P(G/e, L'),$$

where $L'(v) = L(v)$ for all $v \in V(G) \setminus \{v_1, v_2\}$ and $L(u) = L(v_1) \cap L(v_2)$ for $u \in V(G) \setminus V(G/e)$.

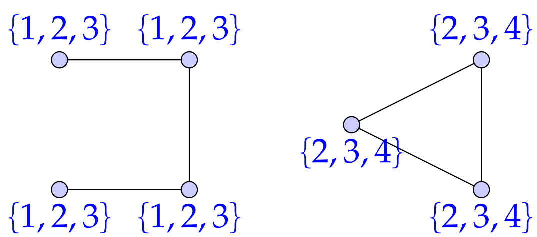
Computing $P(G, L)$



$P(G, L)$ for special assignments L

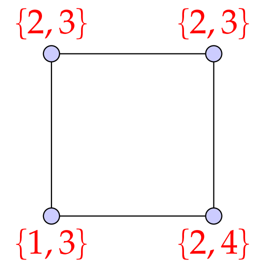
k-assignment L: $|L(v)| = k$ for each vertex v in G.

For a k-assignment L in G, if $L(u) = L(v)$ holds for each edge uv in G, then $P(G, L) = P(G, k)$.



For the above example, $P(G, L) = P(G, 3)$ holds.

$L(u) \neq L(v)$ for some edge uv



A 2-assignment L

$P(C_4, L) = 5$ for this 2-assignment L.

Thus,
$P(C_4, L) > P(C_4, 2) = 2$

List-coloring function $P_l(G, k)$

- For a k-assignment L of G, when $L(u) \neq L(v)$ for some edge uv, it is unknown whether $P(G, L) = P(G, k)$ can hold.
- Introduced by Kostochka and Sidorenko in 1992, the list-coloring function of G, denoted by $P_l(G, k)$, is defined as follows:

$P_l(G, k)$: the minimum value of $P(G, L)$ among all k-assignments L.

Thus,

$$P_l(G, k) = P(G, k)$$
$$\Updownarrow$$
$$P(G, L) \geq P(G, k) \text{ holds for every } k\text{-assignment } L$$

Study on the list-chromatic number $\chi_l(G)$

$\chi_l(G)$ is the least positive integer k such that $P_l(G, k) > 0$. Thus, $\chi_l(G) \geq \chi(G)$.

A well-known result:

Thomassen (1994)

$\chi_l(G) \leq 5$ for each planar graph G.

Conjecture (Vizing, Bollobás, et al), 1970s

For any line graph G, $\chi_l(G) = \chi(G)$.

When does $P_l(G,k) = P(G,k)$ hold?

Trial relations between $P_l(G,k)$ and $P(G,k)$

- By definition, for any integer $k \geq 1$,

$$P_l(G,k) \leq P(G,k).$$

- For any integer $k \geq 1$,

$$P_l(G,k) = P(G,k)$$

$$\Updownarrow$$

for all k-assignments L: $P(G,L) \geq P(G,k)$.

Examples for $P_l(G,k) < P(G,k)$

- For $G = K_{2,4}$,
$$P_l(G,2) = 0 < P(G,2).$$
- For any bipartite graph G with a subgraph $K_{2,4}$:
$$P_l(G,2) = 0 < P(G,2).$$
- For $G = K_{p,p^p}$ and $p \geq 2$,
$$P_l(G,p) = 0 < P(G,2).$$
- For $G = K_{n,n}$, where $n = \binom{2r-1}{r}$ and $2 \leq k \leq r$,
$$P_l(G,k) = 0 < P(G,k).$$

Some open problems

The list-chromatic number $\chi_l(G)$ of G is the smallest number k such that $P_l(G,k) > 0$.

Problems proposed by Thomassen (2009)

1. Does there exist a universal constant α such that, for any graph G and any natural number $k \geq \chi_l(G) + \alpha$, $P_l(G,k) = P(G,k)$ holds?
2. Is it true that if $k = \chi_l(G) > \chi(G)$, then $P_l(G,k) > 1$?
3. Does there exist a graph G and a natural number $k > 2$ such that $P_l(G,k) = 1$?

When does $P_l(G,k) = P(G,k)$ hold?

Problem (Kostochka and Sidorenko, 1992)

When does $P_l(G,k) = P(G,k)$ hold?

$P_l(G,k) = P(G,k)$ holds in the following trivial cases:

- G is an empty graph;
- $P(G,k) = 0$ (i.e., $k < \chi(G)$, in particular $k \le 1$ when G is not empty);
- G is a chordal graph, in particular, G is a tree.

Development

Let G be any simple graph with m edges.

- (Donner, 1992)

$P(G,k) = P_l(G,k)$ holds when k is sufficiently large.

- (Thomassen, 2009)

$P(G,k) = P_l(G,k)$ holds when $k \ge |V(G)|^{10}$.

- (Wang, Quan and Yan, 2017)

$P(G,k) = P_l(G,k)$ holds when $k \ge 1.1346(m-1)$.

Our recent result

Dong and Zhang (2023)

Let G be any simple graph with n vertices and m (≥ 4) edges and k be any integer with $k \ge m-1$. Then, for any k-assignment L of G,

$$P(G,L) - P(G,k) \ge \left((k-m+1)k^{n-3} + c(k-m+3)k^{n-5}\right) \times \sum_{uv \in E(G)} |L(u) \setminus L(v)|,$$

where $c \ge (m-1)(m-3)/24$.

Corollary

Let G be a connected graph with m edges and and k be an integer with $k \ge m-1$.
For any k-assignment L of G, whenever $L(u) \ne L(v)$ for some edge uv in G, $P(G,L) > P(G,k)$ holds.

Hence $P_l(G,k) = P(G,k)$ whenever $k \ge m-1$.

Expressions for $P(G,k)$ and $P(G,L)$

Expression for $P(G,k)$

- Let $G=(V,E)$ be a simple graph with n vertices.
- For any $A\subseteq E$, let $c(A)$ denote the number of components of the spanning subgraph (V,A).
- (Whitney, 1932)

$$P(G,k)=\sum_{A\subseteq E(G)}(-1)^{|A|}k^{c(A)}.$$

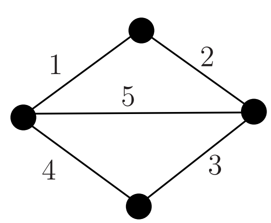
- The above expression can be simplified further by considering broken-cycles.

Broken cycles

Let η be a bijection from $E(G)$ to $\{1,2,\cdots,m\}$, where $m=|E(G)|$.

For any cycle C in G, if e is the edge on C such that $\eta(e)\le\eta(e')$ for each $e'\in E(C)$, then $C\setminus\{e\}$ is called a *broken cycle*.

Let $\mathcal{B}(G)$ be the set of broken cycles in G.



Three broken cycles B_1,B_2,B_3:

$B_1=\{1,2,5\}\setminus\{1\}=\{2,5\}$;

$B_2=\{3,4,5\}\setminus\{3\}=\{4,5\}$;

$B_3=\{1,2,3,4\}\setminus\{1\}=\{2,3,4\}$.

Broken-cycle Theorem

Let $\mathcal{NB}(G)$ denote the set of subsets $A\subseteq E(G)$ such that $B\not\subseteq A$ for every $B\in\mathcal{B}(G)$.

Let $\mathcal{NB}_i(G)$ be the set of $A\in\mathcal{NB}(G)$ with $|A|=i$.

Broken-cycle Theorem (Whitney, 1932)

For any simple graph G of order n,

$$P(G,k)=\sum_{A\in\mathcal{NB}(G)}(-1)^{|A|}k^{c(A)}=\sum_{i=0}^{n-1}(-1)^i|\mathcal{NB}_i(G)|k^{n-i}.$$

Similar expression for $P(G, L)$

- Let $G = (V, E)$ be a simple graph with n vertices, and let L be a k-assignment of G.
- For any $A \subseteq E$, let $c(A)$ denote the number of components of the spanning subgraph (V, A).
- For any k-assignment L of G,

$$P(G, L) = \sum_{A \subseteq E(G)} \left((-1)^{|A|} \beta(G_1, L) \cdots \beta(G_{c(A)}, L) \right),$$

where $G_1, G_2, \cdots, G_{c(A)}$ *are the components of* (V, A) *and*

$$\beta(G_i, L) = \Big| \bigcap_{v \in V(G_i)} L(v) \Big|.$$

Broken-cycle Theorem for $P(G, L)$

For any k-assignment L of G,

$$P(G, L) = \sum_{A \in \mathcal{NB}(G)} \left((-1)^{|A|} \beta(T_1, L) \cdots \beta(T_{c(A)}, L) \right),$$

where $T_1, T_2, \cdots, T_{c(A)}$ *are the components of* (V, A) *and*

$$\beta(T_i, L) = \Big| \bigcap_{v \in V(T_i)} L(v) \Big|.$$

Note that (V, A) is acyclic for any $A \in \mathcal{NB}(G)$.

Lower bound of $P(G, L) - P(G, k)$

$P(G, L) - P(G, k)$

Let $\mathcal{NBF}_i(G)$ be the set of spanning forests F of G with $E(F) \in \mathcal{NB}_i(G)$.

Then

$$P(G, L) - P(G, k) = \sum_{i=1}^{n-1} (-1)^i \sum_{\{T_1, \cdots, T_{n-i}\} \in \mathcal{NBF}_i(G)} \left(\prod_{j=1}^{n-i} \beta(T_j) - k^{n-i} \right).$$

Lower bound of $P(G,L) - P(G,k)$

Dong and Zhang (2023)

For any simple graph G of order n and any k assignment L,

$$P(G,L) - P(G,k) \geq \frac{1}{k} \sum_{e=uv \in E(G)} \left(|L(u) \setminus L(v)| \times Q_\eta(G,e,k) \right),$$

where for each $e \in E(G)$,

$$Q_\eta(G,e,k) = \sum_{\substack{1 \leq i \leq n-1 \\ i \text{ odd}}} \frac{|\mathcal{NB}_i(G,e)|}{i} k^{n-i} - \sum_{\substack{1 \leq i \leq n-1 \\ i \text{ odd}}} |\mathcal{NB}_{i+1}(G,e)| k^{n-i-1}$$

and $\mathcal{NB}_i(G,e)$ is the set of broken-cycle free subsets A of $E(G)$ with $e \in A$ and $|A| = i$.

Properties of the size of $\mathcal{NB}_i(G,e)$

Property 1: For $1 \leq i < m$,

$$|\mathcal{NB}_{i+1}(G,e)| \leq \frac{m-i}{i} |\mathcal{NB}_i(G,e)|.$$

Property 2: $|\mathcal{NB}_1(G,e)| = 1$, and for $i \geq 2$,

$$|\mathcal{NB}_i(G,e)| \geq |\mathcal{NB}_{i-1}(G/e)|.$$

Property 3: For any edge e in G,

$$|\mathcal{NB}_2(G/e)| \geq \frac{(m-1)(m-3)}{8}.$$

Lower bound of $Q_\eta(G,e,k)$

For any $e \in E(G)$ and $k \geq m-1$,

$$\begin{aligned} Q_\eta(G,e,k) &= \sum_{\substack{1 \leq i \leq n-1 \\ i \text{ odd}}} \frac{|\mathcal{NB}_i(G,e)|}{i} k^{n-i} - \sum_{\substack{1 \leq i \leq n-1 \\ i \text{ odd}}} |\mathcal{NB}_{i+1}(G,e)| k^{n-i-1} \\ &\geq \sum_{\substack{1 \leq i \leq n-1 \\ i \text{ odd}}} \frac{|\mathcal{NB}_i(G,e)|}{i} (k-m+i) k^{n-i-1} \\ &\geq (k-m+1)k^{n-2} + \frac{|\mathcal{NB}_3(G,e)|}{3} \cdot (k-m+3)k^{n-4} \\ &\geq (k-m+1)k^{n-2} + \frac{|\mathcal{NB}_2(G/e)|}{3} \cdot (k-m+3)k^{n-4} \\ &\geq (k-m+1)k^{n-2} + \frac{(m-1)(m-3)}{24} (k-m+3)k^{n-4}. \end{aligned}$$

Conclusion

Dong and Zhang (2023)

For any integer $k \geq m-1$ and any k-assignment L,

$$P(G,L) - P(G,k) \geq A \sum_{e=uv \in E(G)} |L(u) \setminus L(v)|,$$

where

$$\begin{aligned} A &= (k-m+1)k^{n-3} + \frac{(m-1)(m-3)(k-m+3)}{24} k^{n-5} \\ &> 0. \end{aligned}$$

Therefore, $P_l(G,k) = P(G,k)$ whenever $k \geq m-1$.

Vertex-coloring in hypergraphs $\mathcal{H}$

Hypergraphs

Question:
What is a hypergraph?

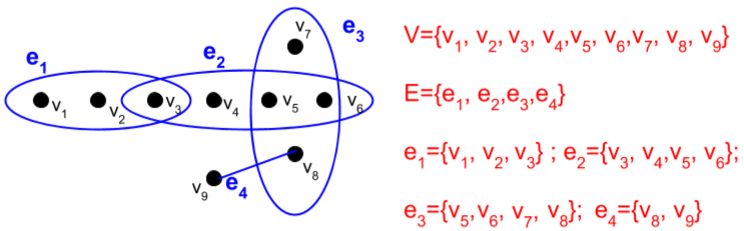


Figure: Hypergraph $\mathcal{H} = (V, E)$

Hypergraphs

Hypergraph

A hypergraph $\mathcal{H} = (V, E)$ consists of finite sets V and E, where

$$E \subseteq \{e \subseteq V : |e| \geq 1\}.$$

If $E \subseteq \{e \subseteq V : |e| = 2\}$, then $\mathcal{H}$ is an ordinary graph.

A hypergraph is also called *a set system* or *a family of subsets* of a universal set V.

Special hypergraphs

- $\mathcal{H} = (V, E)$ is called *r-uniform* if $|e| = r (r \geq 2)$ each edge $e \in E$.
- 2-uniform hypergraphs are ordinary graphs.
- $\mathcal{H}$ is called *linear* if $|e_1 \cap e_2| \leq 1$ for each pair of distinct edges e_1 and e_2 in E.
- A hypergraph is called *Sperner* if $e_1 \not\subseteq e_2$ for each pair of distinct edges e_1 and e_2 in E.

Example

Example

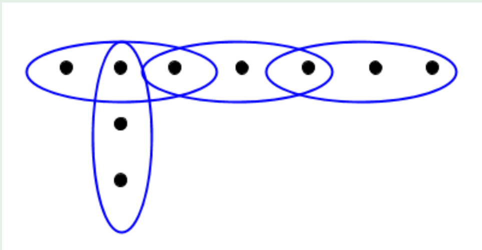


Figure: This hypergraph is 3-uniform, linear and Sperner

k-colouring of a Hypergraph

- For a hypergraph $\mathcal{H} = (V, E)$ and a positive integer k, a (weak) k-colouring of $\mathcal{H}$ is a mapping $f : V \to \{1, 2, \cdots, k\}$ such that $|\{f(u) : u \in e\}| \geq 2$ for each $e \in E$, i.e., for each edge e of $\mathcal{H}$, at least two vertices in e are colored differently by f.
- $\mathcal{H} = (V, E)$ admits a k-colouring if and only if V can be partitioned into k subsets $V_1, V_2, \cdots, V_k$ such that $e \not\subseteq V_i$ for every $e \in E$.

Example

Example

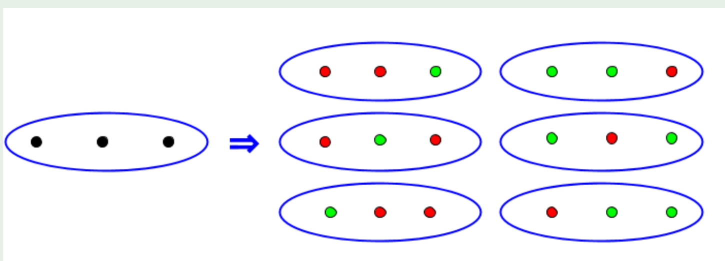


Figure: This hypergraph has 6 different 2-colorings.

For any positive integer k, this hypergraph has exactly $k^3 - k$ different k-colourings.

List-coloring function $P_l(\mathcal{H}, k)$

- Let L be a k-assignment of a hypergraph $\mathcal{H} = (V, E)$.
- A *proper* L-coloring of $\mathcal{H}$ is a mapping $f : V \to \mathbb{N}$ such that $f(v) \in L(v)$ for each $v \in V(G)$ and for each edge $e \in E$,

$$|\{f(v) : v \in e\}| \geq 2.$$

- $P(\mathcal{H}, L)$ denotes the number of proper L-colorings of $\mathcal{H}$.
- $P_l(\mathcal{H}, k)$ is defined to be the minimum value of $P(\mathcal{H}, L)$ over all k-assignments L.

Analogous conclusion on hypergraphs $\mathcal{H}$

Results on r-uniform hypergraphs

Wang, Quan and Yan (2020)

For any r-uniform hypergraph $\mathcal{H}$ with m edges, $P(\mathcal{H},k) = P_l(\mathcal{H},k)$ holds when $k \geq 1.1346(m-1)$.

A parameter $\gamma(\mathcal{H})$

- Let $\mathcal{H}$ be a r-uniform hypergraph with m edges.
- For any edge e in $\mathcal{H}$, let $E_{r-1}(e)$ be the set of edges $e' \in E(\mathcal{H})$ with $|e \cap e'| = r-1$ (i.e., $|e \setminus e'| = 1$).
- Let
$$\gamma(\mathcal{H}) = \max_{e \in E(\mathcal{H})} |E_{r-1}(e)|.$$
- Clearly, $0 \leq \gamma(\mathcal{H}) \leq m-1$.
- $\gamma(\mathcal{H}) = 0$ if and only if $|e \setminus e'| \geq 2$ for every pair of edges e and e'.
- $\gamma(\mathcal{H}) = 0$ if $r \geq 3$ and $\mathcal{H}$ is linear.

An example for $\gamma(\mathcal{H})$

- Let $K_n^{(r)}$ denote the complete r-uniform hypergraph on n vertices.
- $K_n^{(2)}$ is the complete graph K_n.
- Then $m = |E(\mathcal{H})| = \binom{n}{r}$.
- For each edge e in $K_n^{(r)}$, there are exactly $r(n-r)$ edges e' in $K_n^{(r)}$ with $|e \cap e'| = r-1$.
- Thus, $\gamma(\mathcal{H}) = r(n-r)$.
- If $n \geq 11$ and $3 \leq r \leq n-2$, then
$$\frac{\gamma(\mathcal{H})}{m-1} = \frac{n(n-r)}{\binom{n}{r}-1} \leq \frac{1}{3}.$$

Our conclusion

Dong and Zhang (2022+)

Let $\mathcal{H}$ be any r-uniform hypergraph with $m \geq 5$ edges. Then $P(\mathcal{H},k) = P_l(\mathcal{H},k)$ holds when one of the following conditions is satisfied:

1. $k \geq m-1$ if $\gamma(\mathcal{H}) > 0.8(m-1)$;
2. $k \geq 0.6(m-1) + 0.5\gamma(\mathcal{H})$ if $\gamma(\mathcal{H}) \leq 0.8(m-1)$; and
3. $k \geq \frac{1.2(m-1)}{\log(m-1)}$ if $\gamma(\mathcal{H}) = 0$.

Further conclusion

Dong and Zhang (2022+)

Let $\mathcal{H}$ be any uniform hypergraph with m edges. If

$$t = \min_{e_1,e_2 \in E} |e_1 \setminus e_2| \geq 2 \text{ and } m \geq \frac{t^3}{2} + 1,$$

then $P(\mathcal{H},k) = P_l(\mathcal{H},k)$ holds whenever $k \geq \frac{2.4(m-1)}{t \log(m-1)}$.

Problems

Problem

Is there a constant c such that for any graph G with n vertices, $P_l(G,k) = P(G,k)$ holds whenever $k \geq cn$?

Problem

Is there a constant c such that for any graph G with maximum degree Δ, $P_l(G,k) = P(G,k)$ holds whenever $k \geq c\Delta$?

Thanks for your attendance

fengming.dong@nie.edu.sg