Number of 4-dicycles in tournaments with a given score list

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1. Introduction

In this paper, let G be a connected graph with vertex set $V(G) = \{1, 2, \dots, n\}$ and edge set $E(G)$. We consider only simple graphs (i.e., no parallel edges or loops) and simple digraphs (i.e., no parallel arcs or loops). We say that a graph (digraph resp.) G *contains* a graph (digraph resp.) G' if G' is a subgraph (subdigraph resp.) of G, i.e., $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$ ($A(G') \subseteq A(G)$ resp.).

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. If $uv \in A(D)$, where $u, v \in V(D)$, then we say u *dominates* v and denote this by $u \to v$. The subdigraph of D induced by the set of vertices $V \subseteq V(D)$ (set of edges $E \subseteq E(D)$ resp.) is denoted by $\langle V \rangle_D$ ($\langle E \rangle_D$ resp.). The *out-neighbourhood* and *in-neighbourhood* of a vertex $v \in V(D)$ are defined to be $N_D^+(v) = \{x \in V(D) \mid v \to x\}$ and $N_D^-(v) = \{y \in V(D) \mid y \to v\}$ respectively. The *score* s_v or *outdegree* $\deg_D^+(v)$ of a vertex $v \in V(D)$ is defined by $s_v = \deg_D^+(v) = |N_D^+(v)|$. That is, we shall freely interchange between the two notations, s_v and $\deg_D^+(v)$. The *score sequence* (or *score list*) of a tournament T of order n is the ordered n-tuple $(s_1, s_2, \dots, s_n)$. We usually assume that the vertices are labelled in such a way that $s_1 \le s_2 \le \dots \le s_n$. The *co-score* or *indegree* $\deg_D^-(v)$ of a vertex $v \in V(D)$ is defined by $\deg_D^-(v) = |N_D^-(v)|$.

For $k \ge 3$, we denote the k-dicycle (i.e., directed cycle of length k) by C_k. For a digraph F, we use the corresponding small letter, say $f(D)$, to denote the number of copies of F in D. For example, the number of k-dicycles in D is denoted by $c_k(D)$.

Theorem 1.1. *(Kendall and Smith, Szele and Clark (see [9])) Let T be a tournament with score sequence $S = (s_1, s_2, \dots, s_n)$. Then, the number of 3-dicycles in T is*

$$c_3(T) = \binom{n}{3} - \sum_{i=1}^{n} \binom{s_i}{2}. \tag{1.1}$$

Corollary 1.2. *If T is a tournament of order n, then*

$$c_3(T) \le \begin{cases} \frac{1}{24} n(n+1)(n-1), & \text{if } n \text{ is odd,} \\ \frac{1}{24} n(n+2)(n-2), & \text{if } n \text{ is even.} \end{cases}$$

Equality holds only for regular and near-regular tournaments in the respective cases.

The expression (1.1) tells us that all tournaments with the same score list has the same number of 3-dicycles. The next question follows naturally. Does all tournaments with the same score list have the same number of 4-dicycles?

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Motivation for *fixed* score list

One of the early problems concerning arc reversals is: Given two tournaments of the same order, is it possible to obtain one from the other by a sequence of a prescribed type of arc reversals?

Theorem 1.3. *(Ryser [11], Waldrop [12]) Two tournaments have the same score sequence if and only if one can be obtained from the other via a sequence of arc reversals of C_3.*

Example 1.4. The tournaments T and T' in Figure 1 have the same score list $(1, 2, 2, 2, 3)$. By Ryser's Theorem, T' can be obtained from T via a sequence of arc reversals of C_3.

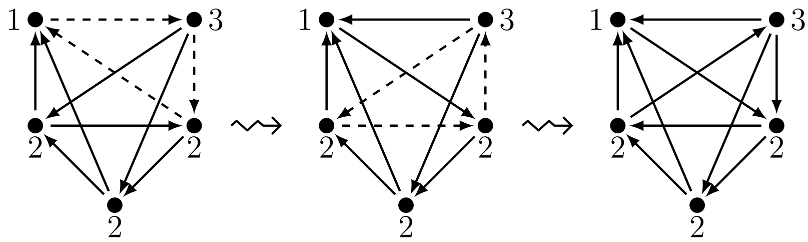


Figure 1: C_3-reversals for 3 tournaments with the same score sequence $(1, 2, 2, 2, 3)$.

Later, Waldrop [12] gave an independent proof of Theorem 1.3 and further established two results in which "C_3" is replaced by "C_4" and by "C_5".

Theorem 1.5. *(Beineke and Moon [4]) Two bipartite tournaments have the same score lists if and only if one can be obtained from the other via a sequence of arc reversals of C_4.*

For any graph G, let $\mathcal{L}_G$ denote the set of lengths of all cycles (chordless cycles resp.) in G, i.e., $\mathcal{L}_G = \{l \mid G \text{ contains a chordless } l\text{-cycle}\}$. Let G be a connected graph and H_i, $i = 1, 2, \dots, n$, be graphs which are pairwise vertex-disjoint. The *composition* $G[H_1, H_2, \dots, H_n]$ is the graph with vertex set $V^* = \bigcup_{i=1}^n V(H_i)$ and edge set $E^* = \bigcup_{i=1}^n E(H_i) \cup \{uv \mid u \in V(H_j), v \in V(H_k), jk \in E(G)\}$. We denote the complete graph of order n and its complement by K_n and $\bar{K}_n$ respectively.

Using a generalised notion of orientations having the "same score list", Wong and Tay [13, 14] extended Theorems 1.3 and 1.5 to orientations of composition of graphs.

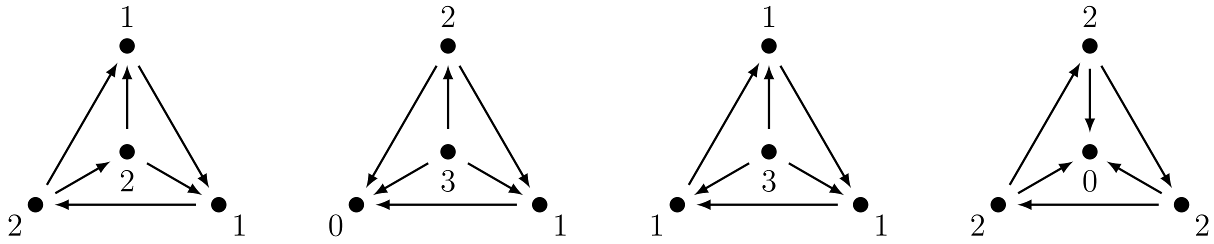
Theorem 1.6. *(Wong and Tay [13]) Let G be a graph and p_i, $i = 1, 2, \dots, n$, be positive integers. Two orientations of $G[\bar{K}_{p_1}, \bar{K}_{p_2}, \dots, \bar{K}_{p_n}]$ have the same score list if and only if one can be obtained from the other via a sequence of arc reversals of C_i, where $i \in \{4\} \cup \mathcal{L}_G$.*

Theorem 1.7. *(Wong and Tay [14]) Let G be a graph and p_i, $i = 1, 2, \dots, n$, be positive integers.Two orientations of $G[K_{p_1}, K_{p_2}, \dots, K_{p_n}]$ have the same score list if and only if one can be obtained from the other via a sequence of arc reversals of C_i, where $i \in \{3\} \cup \mathcal{L}_G$.*

Theorem 1.8. *(Wong and Tay [14]) Let G and H_i, $i = 1, 2, \dots, n$, be graphs and let $\mathcal{L} = \mathcal{L}_G \cup \bigcup_{i=1}^n \mathcal{L}_{H_i}$. Two orientations of $G[H_1, H_2, \dots, H_n]$ have the same score list if and only if one can be obtained from the other via a sequence of arc reversals of C_i, where $i \in \{3, 4\} \cup \mathcal{L}$.*

2. Number of 4-dicycles

We denote the four non-isomorphic tournaments of order 4 as X_1, X_2, X_3, X_4 (see Figure 2).



(a) X_1 $(1,1,2,2)$ Two C_3 — (b) X_2 $(0,1,2,3)$ No C_3 — (c) X_3 $(1,1,1,3)$ One C_3 — (d) X_4 $(0,2,2,2)$ One C_3

Figure 2: All non-isomorphic tournaments of order 4.

Remark 2.1. Note that any tournament of order 4 contains at most a 4-dicycle, and contains exactly one if and only if it is X_1.

Theorem 2.2. *(Beineke and Harary [3], Alspach and Tabib [1]) Let T be a tournament with score sequence $S = (s_1, s_2, \ldots, s_n)$. Then, the number of 4-dicycles in T is*

$$c_4(T) = \binom{n}{4} - \sum_{i=1}^{n} \binom{s_i}{3} - \sum_{i=1}^{n} \binom{n-1-s_i}{3} + \sum_{i=1}^{n} \sum_{v_j \in O(v_i)} \binom{r_{i,j}}{2}, \tag{2.1}$$

where $r_{i,j}$ is the score of the vertex v_j in $\langle O_T(v_i) \rangle_T$.

Theorem 2.3. *(Alspach and Tabib [1]) Let T be a tournament with scores $s_1, s_2, \ldots, s_n$. Then,*

$$\binom{n}{4} - \min\left\{ \sum_{i=1}^{n} \left[\binom{s_i}{3} + (s_i)^* \right], \sum_{i=1}^{n} \left[\binom{n-s_i-1}{3} + (n-s_i-1)^* \right] \right\} \leq c_4(T),$$

$$\text{where } (m)^* = \begin{cases} \frac{1}{24} m(m+1)(m-1), & \text{if } m \text{ is odd}, \\ \frac{1}{24} m(m+2)(m-2), & \text{if } m \text{ is even}. \end{cases}$$

and

$$c_4(T) \leq \binom{n}{4} - \max\left\{ \sum_{i=1}^{n} \binom{s_i}{3}, \sum_{i=1}^{n} \binom{n-s_i-1}{3} \right\}.$$

Proof: By noting the number of vertices with score 0 and 3 in X_i, $i = 1, 2, 3, 4$, it is straightforward to derive that

$$\left.\begin{aligned} \binom{n}{4} &= x_1(T) + x_2(T) + x_3(T) + x_4(T), \\ \sum_{i=1}^{n} \binom{n-1-s_i}{3} &= x_2(T) + x_4(T), \\ \sum_{i=1}^{n} \binom{s_i}{3} &= x_2(T) + x_3(T). \end{aligned}\right\} \tag{2.2}$$

These yield

$$x_1(T) = \binom{n}{4} - \sum_{i=1}^{n} \binom{s_i}{3} - x_3(T), \text{ and} \tag{2.3}$$

$$x_1(T) = \binom{n}{4} - \sum_{i=1}^{n} \binom{n-1-s_i}{3} - x_4(T). \tag{2.4}$$

The upper bound follows since zero is the smallest that $x_3(T)$ and $x_4(T)$ can be. The lower bound follows by making $x_3(T)$ and $x_4(T)$ as large as possible. This happens precisely when every outset or every inset of the vertices has the maximum number of 3-dicycles contained in it. □

Remark 2.4. Note that (2.3) and (2.4) are also expressions of the number of 4-dicycles in T since $x_1(T) = c_4(T)$.

Corollary 2.5. *(Alspach and Tabib [1]) If T is a tournament of order n, then*

$$c_4(T) \leq \begin{cases} \frac{1}{48}n(n+1)(n-1)(n-3), & \text{if } n \text{ is odd,} \\ \frac{1}{48}n(n+2)(n-2)(n-3), & \text{if } n \text{ is even.} \end{cases}$$

Equality is achieved by regular and near-regular tournaments in the respective cases.

Corollary 2.6. *(Alspach and Tabib [1]) Let $\mathcal{T}$ be a class of tournaments with the same score list. If $T, T' \in \mathcal{T}$ and $\Delta x_i = x_i(T) - x_i(T')$, $i = 1, 2, 3, 4$, then $\Delta x_1 = \Delta x_2 = -\Delta x_3 = -\Delta x_4$.*

Proof: By further noting the number of vertices with score 2 in X_i, $i = 1, 2, 3, 4$, we have

$$\sum_{i=1}^{n} \binom{s_i}{2} \binom{n-1-s_i}{1} = 2x_1(T) + x_2(T) + 3x_4(T) \text{ and} \tag{2.5}$$

Using (2.2) and (2.5), it follows for any $T, T' \in \mathcal{T}$ that

$$\begin{aligned} 0 &= \Delta x_1 + \Delta x_2 + \Delta x_3 + \Delta x_4, \\ 0 &= \Delta x_2 + \Delta x_4, \\ 0 &= 2\Delta x_1 + \Delta x_2 + 3\Delta x_4 \text{ and} \\ 0 &= \Delta x_2 + \Delta x_3. \end{aligned}$$

The result follows by solving this system of linear equations.

Corollary 2.7. *Let $\mathcal{T}$ be a class of tournaments with a given score sequence. For any $T \in \mathcal{T}$, the following are equivalent:*
(1) T contains the minimum (maximum resp.) copies of X_1.
(2) T contains the minimum (maximum resp.) copies of X_2.
(3) T contains the maximum (minimum resp.) copies of X_3.
(4) T contains the maximum (minimum resp.) copies of X_4.

3. Maximising $c_4(T)$

By Corollary 2.7, if $x_3(T) = 0$ or $x_4(T) = 0$, then T has the maximum number of 4-dicycles in a class of tournaments with a given score list. Alspach and Tabib [1] characterised all the score lists for which there exists a tournament T of order n with $x_3(T) = x_4(T) = 0$, in which case $x_1(T) = \binom{n}{4} - \sum_{i=1}^{n} \binom{s_i}{3}$ and $x_2(T) = \sum_{i=1}^{n} \binom{s_i}{3}$.

In all computations with subscripts, they are to be reduced modulo n using the residues $\{1, 2, \ldots, n\}$. Given a labelling $u_1, u_2, \ldots, u_n$ of vertices of the n-tournament T, we say that u_i is *domination oriented* if u_i dominates $u_{i+1}, u_{i+2}, \ldots, u_{i+s_i}$ and is dominated by $u_{i+s_i+1}, u_{i+s_i+2}, \ldots, u_{i-1}$, where s_i denotes the score of u_i. The tournament T is said to be *domination orientable* if there is a labelling of its vertices so that every vertex is domination oriented.

Theorem 3.1. *(Alspach and Tabib [1]) The tournament T is domination orientable if and only if $\langle N^+(u)\rangle$ and $\langle N^-(u)\rangle$ are transitive tournaments for every vertex u of T.*

Using the previous theorem, they characterised the score lists that have tournaments all of whose outsets and insets are transitive. A score list $s_1 \le s_2 \le \ldots \le s_n$ is said to be balanced if $s_i + s_{n-i+1} = n - 1$ for $i = 1, 2, \ldots, \lceil \frac{n}{2} \rceil$. Notice that when n is odd, then $s_{(n+1)/2} = (n-1)/2$.

Theorem 3.2. *Let $S = (s_1, s_2, \ldots, s_n)$ be a sequence of nonnegative integers satisfying $s_1 \le s_2 \le \ldots s_n$. Then there is a domination orientable tournament T with score lists S if and only if either*
(i) $S = (0, 1, 2, \ldots, n-1)$, or
(ii) S is balanced, S is strong tournament realizable and for each m satisfying $n - s_n - 1 \le m \le s_n$ there is some s_i equal to m.

Corollary 3.3. *Let $S = (s_1, s_2, \ldots, s_n)$ be a score list satisfying the conditions of Theorem 3.2. Then the maximum number of 4-dicycles contained in any tournament with score-list S is*

$$c_4(T) = \binom{n}{4} - \sum_{i=1}^{n} \binom{s_i}{3}$$

and this bound is attained.

The minimisation problem of $c_4(T)$ (i.e., achieving the lower bound of Theorem 2.3) is more difficult as one needs each outset or inset to induce a regular or near-regular tournament (see [1, 5]).

4. Questions for future research

1. Generalise the expressions (1.1) and (2.1) for multipartite tournaments or for other values of k in C_k, i.e., in a class of multipartite tournaments T with the same score list, what is an expression of the number of 3-dicycles (or 4-dicycles) in T? Hence, determine the maximum (or minimum) number of 3-dicycles (or 4-dicycles) in T.

2. Is it possible to modify the notion of domination orientable for tournaments to settle the cases $x_3(T) = 0$ and $x_4(T) = 0$ respectively?

3. Domination orientable tournaments were considered by Berman [6] in a 5-dicycle maximisation problem. Generalise the notion of domination orientable to multipartite tournament. i.e., what significance does the generalised notion have with respect to the number of k-dicycles?

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