Number of quasi-kernels in digraphs

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Gregory Gutin, Khee Meng Koh, Eng Guan Tay and Anders Yeo (2004) On the number of quasi-kernels in digraphs. *Journal of Graph Theory 46*(1), 48–56. DOI: 10.1002/jgt.10169

Abstract

A vertex set X of a digraph $D = (V, A)$ is a *kernel* if X is independent (i.e., all pairs of distinct vertices of X are non-adjacent) and for every $v \in V$-X there exists $x \in X$ such that $vx \in A$. A vertex set X of a digraph $D = (V, A)$, is a *quasi-kernel* if X is independent and for every $v \in V$- there exist $w \in V$-X, $x \in X$ such that either $vx \in A$ or vw, $wx \in A$. In 1974, Chvátal and Lovász proved that every digraph has a quasi-kernel. In 1996, Jacob and Meyniel proved that if a digraph D has no kernel, then D contains at least three quasi-kernels. We characterize digrapl with exactly one and two quasi-kernels, and, thus, provide necessary and sufficient condition a digraph to have at least three quasi-kernels. In particular, we prove that every strong digrap order at least three, which is not a 4-cycle, has at least three quasi-kernels.

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Definition 1
A vertex set X of a digraph $D = (V, A)$ is a kernel if X is independent and for every $v \in V - X$, there exists $x \in X$, such that $vx \in A$.

Definition 2
A vertex set X of a digraph $D = (V, A)$ is a quasi-kernel if X is independent and for every $v \in V - X$, there exist $w \in V - X$, $x \in X$, such that either $vx \in A$ or vw, $wx \in A$.

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Theorem (Chvátal & Lovász, 1974)
Every digraph has a quasi-kernel.

Theorem (Jacob & Meyniel, 1996)
If a digraph D has no kernel, then D has at least 3 quasi-kernels.

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Lemma 2.1 *Let x be a vertex in a digraph D. If x is a non-sink, then D has a quasi-kernel not including x.*

Proof: Let $y \in N^+[x] - \{x\}$ be arbitrary. If $N^-[y] = V(D)$, then y is the required quasi-kernel. If $N^-[y] \neq V(D)$, let Q' be a quasi-kernel in $D - N^-[y]$. If y dominates a vertex in Q', then Q' is a quasi-kernel in D, which does not contain x. If y does not dominate a vertex in Q', then $Q' \cup \{y\}$ is a quasi-kernel in D, which does not include x.□

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Theorem 2.2 *A digraph D has only one quasi-kernel if and only if D has a sink and every non-sink of D dominates a sink of D. If a digraph D has only one quasi-kernel Q, then Q is a kernel and consists of the sinks of D.*

Proof: Assume that D has a sink and every non-sink of D dominates a sink of D. Let S be the set of sinks in D. To see that S is a unique quasi-kernel of D, it is enough to observe that every sink must be in a quasi-kernel.

Let D have only one quasi-kernel Q. To see that Q is the set of sinks in D, observe that Q contains all sinks in D and, by Lemma 2.1, Q does not have non-sinks. If x is a non-sink and x does not dominate a vertex in Q, then $Q \cup \{x\}$ is another quasi-kernel of D, a contradiction. Thus, we have proved that D has a sink and every non-sink of D dominates a sink of D. □

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In view of Theorem 2.2, the following assertion is a strengthening of the Jacob-Meyniel theorem for the case of digraphs with no sinks.

Theorem 2.3 *Let D be a digraph with no sink. Then D has precisely two quasi-kernels if and only if D has an induced 4-cycle or 2-cycle, C, such that no vertex of C dominates a vertex in $D - V(C)$ and every vertex in $D - V(C)$ dominates at least two adjacent vertices in C.*

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Theorem 2.5 *A strong digraph D of order at least three has at least three quasi-kernels, unless D is $\vec{C}_4$.*

Proof: Immediate from the previous theorems, Theorems 2.2 and 2.3. □

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Theorem 2.6 *Let D be a digraph, S the set of sinks in D, R the set of vertices that have an arc into S, and $H = D - S - R$. Then D has precisely two quasi-kernels, if and only if one of the following holds:*

(a) There is a 2-cycle C in H such that at most one of the vertices in C has an arc into R, no vertex of C dominates a vertex in $H - V(C)$, and every vertex in $H - V(C)$ dominates both vertices in C.

(b) There is an induced 4-cycle, C, in H such that no vertex of C dominates a vertex in $D - V(C)$ and every vertex in $H - V(C)$ dominates two adjacent vertices in C.

(c) The digraph H has at least two vertices. There is a vertex x in H such that no vertex of H is dominated by x, all the vertices of $H - x$ dominate x, i.e., $(V(H) - \{x\}, x) = (V(H) - \{x\}) \times \{x\}$, and there is a kernel Q in $H - x$, consisting only of sinks in $H - x$. Moreover, there is no arc from Q to R.

(d) The digraph H has exactly one vertex and this vertex dominates a vertex in R.

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3 Disjoint quasi-kernels

If a digraph D has a sink x, then every quasi-kernel in D must contain x. Hence, a digraph with sinks has no disjoint quasi-kernels. However, one may suspect that every digraph with no sink has a pair of disjoint quasi-kernels. By Lemma 2.1, this is true for digraphs with exactly two quasi-kernels: see the first paragraph in the proof of Theorem 2.3. One can show that this is also true for every digraph which possesses a quasi-kernel of cardinality at most two.

Unfortunately, in general, the above claim does not hold. Consider the following construction suggested to us by the referee. Let T be a tournament having the property that for every pair x, y of vertices there exists a vertex z such that $x \rightarrow z$ and $y \rightarrow z$. (The existence of such tournaments was first proved by Erdős [4], see also Section 1.2 in [1]. It was shown by Graham and Spencer [5] that some quadratic residue tournaments are such tournaments, see also Section 9.1 in [1].) Extend T to a digraph D by adding, for every vertex x in T, a new vertex x' together with the arc $x'x$.

Clearly, D has no sink and every quasi-kernel of D contains exactly one vertex in T. If Q_x and Q_y are a pair of quasi-kernels of D containing the vertices x and y, respectively, then they are not disjoint because they both have to contain z', where $x \rightarrow z$ and $y \rightarrow z$.

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Scott Heard & Jing Huang (2008). Disjoint quasi-kernels in digraphs

Abstract: A *quasi-kernel* in a digraph is an independent set of vertices such that any vertex in the digraph can reach some vertex in the set via a directed path of length at most two. Chvátal and Lovász proved that every digraph has a quasi-kernel. Recently, Gutin et al. raised the question of which digraphs have a pair of disjoint quasi-kernels. Clearly, a digraph has a pair of disjoint quasi-kernels cannot contain *sinks*, that is, vertices of out-degree zero, as each such vertex is necessarily included in a quasi-kernel. However, there exist digraphs which contain neither sinks nor a pair of disjoint quasi-kernels. Thus, containing no sinks is not sufficient in general for a digraph to have a pair of disjoint quasi-kernels. In contrast, we prove that, for several classes of digraphs, the condition of containing no sinks guarantees the existence of a pair of disjoint quasi-kernels. The classes contain semicomplete multipartite, quasi-transitive, and locally semicomplete digraphs. J Graph Theory 58: 251–260, 2008

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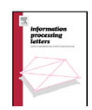

A note on quasi-kernels in digraphs

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ABSTRACT

We describe a simple linear time algorithm to construct a quasi-kernel in a digraph and to find three quasi-kernels in digraphs without a kernel (giving constructive proofs of known results of Chvátal and Lovász, or Jacob and Meyniel). However, we show that it is NP-complete to decide if there is a quasi-kernel containing a specified vertex in a given digraph. The algorithm provides also a simple proof of the characterization of digraphs with only two quasi-kernels given by Gutin, Koh, Tay and Yeo.

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Note

At least three minimal quasi-kernels

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If D is a digraph, then $K \subseteq V(D)$ is a quasi-kernel of D if K is independent and for each $y \in V(D) - K$ there is $x \in K$ such that the directed distance from y to x is less than three. Note that any independent superset of a quasi-kernel is a quasi-kernel. Jacob and Meyniel have given a sufficient condition for a digraph to have at least three quasi-kernels, however these quasi-kernels need not be minimal. We give sufficient conditions for a digraph to have at least three minimal quasi-kernels.

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A Result on the Small Quasi-Kernel Conjecture

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Abstract

Any directed graph $D = (V(D), A(D))$ in this work is assumed to be finite and without self-loops. A source in a directed graph is a vertex having at least one ingoing arc. A quasi-kernel $Q \subseteq V(D)$ is an independent set in D such that every vertex in $V(D)$ can be reached in at most two steps from a vertex in Q. It is an open problem whether every source-free directed graph has a quasi-kernel of size at most $|V(D)|/2$, a problem known as the small quasi-kernel conjecture (SQKC). The aim of this paper is to prove the SQKC under the assumption of a structural property of directed graphs. This relates the SQKC to the existence of a vertex $u \in V(D)$ and a bound on the number of new sources emerging when u and its out-neighborhood are removed from D. The results in this work are of technical nature and therefore additionally verified by means of the Coq proof-assistant.

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References

[1] Results on the Small Quasi-Kernel Conjecture. J. Ai, S. Gerke, G. Gutin, A. Yeo, and Y. Zhou. arXiv:2207.12157, 2022.

[2] N. Alon. Open Problems. *Fete of Combinatorics and Computer Science*. 355–365, 2010, Springer.

[3] V. Chvátal and L. Lovász. Every directed graph has a semi-kernel. *Hypergraph Seminar*, 175–175, 1974, Springer.

[4] Towards a theory of domination in graphs. E. Cockayne and S. Hedetniemi. *Networks*, **7**(3):247–261, 1977.

[5] G. Gutin, K. Koh, E. Tay, and A. Yeo. On the number of quasi-kernels in digraphs. *Journal of Graph Theory*, **46**(1):48–56, 2004.

[6] S. Heard and J. Huang. Disjoint quasi-kernels in digraphs. *Journal of Graph Theory*, **58**(3):251–260, 2008.

[7] A. van Hulst. Kernels and Small Quasi-Kernels in Digraphs. arXiv:2110.00789, 2021.

[8] H. Jacob and H. Meyniel. About quasi-kernels in a digraph. *Discrete Mathematics*, **154**(1-3):279–280, 1996.

[9] A. Kostochka, R. Luo and S. Shan. Towards the small quasi-kernel conjecture. *Electronic Journal in Combinatorics*, **29**(3):1-6.

[10] M. Richardson. On weakly ordered systems. *Bulletin of the American Mathematical Society*, **52**:113–116, 1946.

[11] Small quasi-kernels in directed graphs. http://lemon.cs.elte.hu/egres/open/Small_quasi-kernels_in_directed_graphs.

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