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Expressions for Matching Polynomials*

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Abstract

This paper studies representations of the matching polynomial $m(G,x)$ of a simple graph G. It is well known that $m(G,x) = \det(xI_n - A)$ if G does not contain cycles, where n is the order of G and A is the adjacency matrix of G. Aihara in 1979 posed an open problem: does every graph G of order n have a Hermitian matrix H such that $m(G,x) = \det(xI_n - H)$, where a Hermitian matrix $(a_{j,k})$ is a square matrix with the property that each entry is a complex number and $a_{j,k}$ and $a_{k,j}$ are conjugates for all j,k? He solved this problem for unicyclic graphs. Yan, Yeh and Zhang recently found solutions for graphs which contain a small number of odd cycles but no even cycles. In this paper we solve this problem for any graph in which all cycles are edge-disjoint. For a general graph G, Godsil and Gutman in 1978 and Yan, Yeh and Zhang in 2005 established different expressions for $m(G,x)$ in terms of $\det(xI_n - H)$ for some families of matrices H. This paper generalizes their results and greatly simplifies the computation of $m(G,x)$.

Keywords: graph, adjacency matrix, matching polynomial, characteristic polynomial

1 Introduction

In this paper we consider simple graphs only. For any graph G, let $V(G)$, $E(G)$ and $v(G)$ be its vertex set, edge set and order (i.e., $v(G) = |V(G)|$). If it is not mentioned elsewhere, we always assume that G is a graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. A *matching* of G is a subset M of E such that each vertex of G is incident with at most one edge in M. For any integer $k \geq 0$, let $\phi_k(G)$ denote the number of matchings M of G with $|M| = k$. Thus $\phi_0(G) = 1$, $\phi_1(G) = |E|$ and $\phi_k(G) = 0$ if $k \notin \{0, 1, 2, \cdots, \lfloor v(G)/2 \rfloor\}$. In particular, let ϕ_G be the number of perfect matchings of G, i.e., $\phi_G = \phi_{v(G)/2}(G)$.

The *matching polynomial* of G (see [4]), denoted by $m(G,x)$, is defined as follows:

$$m(G,x) = \sum_{k=0}^{\lfloor v(G)/2 \rfloor} (-1)^k \phi_k(G) x^{v(G)-2k}. \tag{1.1}$$

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