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An elementary proof of the identity $\cot\theta = \frac{1}{\theta} + \sum_{k=1}^{\infty} \frac{2\theta}{\theta^2 - k^2\pi^2}$

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Abstract

This paper gives an elementary proof of the famous identity

$$\cot\theta = \frac{1}{\theta} + \sum_{k=1}^{\infty} \frac{2\theta}{\theta^2 - k^2\pi^2}, \quad \theta \in \mathbb{R}\backslash\pi\mathbb{Z}.$$

Using nothing more than freshman calculus, the present proof is far simpler than many existing ones. This result also leads directly to the Euler's and Neville's identities, as well as the identity $\zeta(2) := \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

1 Introduction

The famous identity,

$$\pi\cot(\pi z) = \sum_{\nu=-\infty}^{\infty} \frac{1}{z+\nu} = \frac{1}{z} + \sum_{\nu=1}^{\infty} \frac{2z}{z^2-\nu^2}, \quad z \in \mathbb{C}\backslash\mathbb{Z} \tag{1}$$

was probably known to Euler, who later in 1770 presented a more general (and more complicated) form

$$\frac{\pi}{n}\cdot\frac{\cos\frac{\pi(w-z)}{2n}}{\sin\frac{\pi(w+z)}{2n} - \sin\frac{\pi(w-z)}{2n}} = \frac{1}{z} + \sum_{\nu=1}^{\infty}\left(\frac{2w}{(2\nu-1)^2n^2 - w^2} - \frac{2z}{(2\nu)^2n^2 - z^2}\right)$$

in [3], where $n \in \mathbb{Z}^+$ and $w,\ z \in \mathbb{C}$. Equation (1) which is also known as the *partial fraction representation* of $\pi\cot(\pi z)$ can be shown to hold for non-integer complex numbers z, using complex analysis [10, 9, 2]. In particular, [2] is the first to put in print a proof that employs the so-called *Herglotz's trick*, thereby simplifying the proof given in [9]. For an excellent historical account of this identity and more, the reader is referred to [8].

The real version of (1), expressed as

$$\cot x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2x}{x^2 - k^2\pi^2}, \quad x \in \mathbb{R}\backslash\pi\mathbb{Z}, \tag{2}$$

is already an interesting identity in itself, attracting numerous elegant proofs using real analysis (see [7, 5]). Unfortunately, many of these proofs are quite involved and thus not readily accessible by beginners in calculus.

This paper presents an elementary and simpler proof of (2), using no more than freshman calculus. Because of its lighter mathematical overhead, it is hoped that the material herein can be of educational value in the following sense. Instructors of calculus courses can make suitable adaptations of this material for classroom teaching and discussion, design of investigative activities and enrichment experiences. The paper is also written with teacher training purposes in mind, i.e., to supply trainee teachers with pedagogical content knowledge relevant to the teaching of calculus.

We organize the paper as follows. In Section 2, we establish some elementary inequalities essential in the proof of the main result. Our proof of Equation(2) appears in Section 3. The section is written in a self-contained manner, i.e., each of the results we employ can be proven with background knowledge acquired from a first year undergraduate Calculus course. In doing so, we demonstrate the usefulness of the crucial results commonly taught in such a course. We intentionally and completely bypass *convergence theorems*; whenever an interchange of limits, infinite sums and (differential or integral) operators is invoked, there is always an elementary alternative. Section 4 showcases some interesting applications of the identity(2). This serves resource materials that can be employed in the design of classroom activities for a freshman course in Calculus. Throughout the paper, we denote the set of integers, integral multiples of π, real numbers and complex numbers by $\mathbb{Z}, \pi\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ respectively.

2 Some inequalities

We start with two basic inequalities commonly encountered in Calculus.

Proposition 2.1. *If $t \in (0, \pi)$, then $t - \frac{t^3}{3!} \le \sin t \le t - \frac{t^3}{3!} + \frac{t^5}{5!}$.*

Proof. We establish only the lower bound, leaving the upper one as an exercise for the diligent reader. Consider $g(t) = \sin t - \left(t - \frac{t^3}{3!}\right)$. Since $g''(t) = -\sin t + t \ge 0$, it follows from $g'(0) = 0$ that $g'(t) = \cos t - \left(1 - \frac{t^2}{2!}\right) \ge 0$. Because $g(0) = 0$, it then follows that $g(t) \ge 0$. □

Proposition 2.2. *If $t \in (0, \frac{\pi}{2})$, then $\frac{t}{\sin t} \leq \frac{\pi}{2}$.*

Proof. This follows since $f(t) = \frac{\sin t}{t}$ is monotone decreasing on $(0, \frac{\pi}{2})$. $\square$

These results lead to the following handy inequality:

Lemma 2.3. *If $t \in (0, \frac{\pi}{2})$, then $0 < -\frac{1}{t^2} + \csc^2 t < 1$.*

Proof. Since $0 < \sin t < t$ for all $t \in (0, \frac{\pi}{2})$, we have $0 < \frac{1}{t^2} < \csc^2 t$ so that $0 < -\frac{1}{t^2} + \csc^2 t$. Now, by Proposition 2.2,

$$-\frac{1}{t^2}+\csc^2 t = \frac{(t-\sin t)(t+\sin t)}{t^2 \sin^2 t} \leq \frac{(t-t+\frac{t^3}{3!})(2t)}{t^2 \sin^2 t} = \frac{1}{3}\left(\frac{t}{\sin t}\right)^2 \leq \frac{1}{3}\cdot\frac{\pi^2}{4} < 1.$$

$\square$

3 Partial fraction representation of $\cot\theta$

The topic of partial fractions is key to many national curricula for pre-university or high school mathematics (e.g., [6, 4]). Apart from its usual role in series expansion and antiderivatives of rational functions, its connection with other elementary functions is seldom mentioned.

In this section, our main aim is to obtain the following partial fraction representation for $\cot\theta$:

Theorem 3.1. *For $\theta \in \mathbb{R}\backslash\pi\mathbb{Z}$, it holds that*

$$\cot\theta = \frac{1}{\theta} + \sum_{k=1}^{\infty} \frac{2\theta}{\theta^2 - k^2\pi^2}. \tag{3}$$

Such a partial fraction representation must necessarily involve an infinite sum since any finite partial fraction amounts to only a rational function, *not* a trigonometric one. Because an infinite series can be seen as the limit of a recursion, it deems fit to initiate our proof from a common-looking trigonometric identity which somewhat 'recursively' defines $\cot\theta$.

Proposition 3.2. *For any $x \in (0, \pi)$, the following identity holds:*

$$\cot x = \frac{1}{2}\left(\cot\frac{x}{2} - \tan\frac{x}{2}\right).$$

Proof. Take reciprocals in the double angle formula $\tan x = \frac{2\tan x}{1-\tan^2 x}$. $\square$

With a recursive mind-set, we apply the preceding proposition thrice to obtain

$$\begin{aligned}
\cot\theta &= \frac{1}{2}\left(\cot\frac{\theta}{2} - \tan\frac{\theta}{2}\right)\\
&= \frac{1}{2}\left(\frac{1}{2}\cot\frac{\theta}{4} - \frac{1}{2}\tan\frac{\theta}{4} - \tan\frac{\theta}{2}\right)\\
&= \frac{1}{2^2}\left(\cot\frac{\theta}{2^2} - \tan\frac{\theta}{2^2}\right) - \frac{1}{2}\cot\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\\
&= \frac{1}{2^2}\left(\cot\frac{\theta}{2^2} - \tan\frac{\theta}{2^2}\right) - \frac{1}{2}\cdot\frac{1}{2}\left(\cot\frac{\pi-\theta}{2^2} - \cot\frac{\pi+\theta}{2^2}\right)\\
&= \frac{1}{2^2}\left(\cot\frac{\theta}{2^2} - \tan\frac{\theta}{2^2}\right) + \frac{1}{2^2}\left(\cot\frac{\theta+\pi}{2^2} + \cot\frac{\theta-\pi}{2^2}\right).
\end{aligned}$$

A straightforward induction on $n \geq 2$ then yields the generalization below.

Proposition 3.3. *If $\theta \in (0, \pi)$ and $n \geq 2$ is an integer, then*

$$\cot\theta = \frac{1}{2^n}\left(\cot\frac{\theta}{2^n} - \tan\frac{\theta}{2^n}\right) + \frac{1}{2^n}\sum_{k=1}^{2^{n-1}-1}\left(\cot\frac{\theta+k\pi}{2^n} + \cot\frac{\theta-k\pi}{2^n}\right).$$

The well-worn fact that $\lim_{\alpha\to 0}\frac{\alpha}{\sin\alpha} = 1$ justifies that

Proposition 3.4. *If $\theta \in (0, \pi)$, then*

$$\lim_{n\to\infty}\frac{1}{2^n}\left(\cot\frac{\theta}{2^n} - \tan\frac{\theta}{2^n}\right) = \frac{1}{\theta}.$$

Based on Propositions 3.3 and 3.4, it becomes evident that to achieve (3) one must prove that:

Theorem 3.5. *If $\theta \in (0, \pi)$, then*

$$\lim_{n\to\infty}\left\{\frac{1}{2^n}\sum_{k=1}^{2^{n-1}-1}\left(\cot\frac{\theta+k\pi}{2^n} + \cot\frac{\theta-k\pi}{2^n}\right) - \sum_{k=1}^{2^{n-1}-1}\frac{2\theta}{\theta^2-k^2\pi^2}\right\} = 0.$$

Proof. For $n \geq 2$, we have:

$$\begin{aligned}
&\frac{1}{2^n}\sum_{k=1}^{2^{n-1}-1}\left(\cot\frac{\theta+k\pi}{2^n} + \cot\frac{\theta-k\pi}{2^n}\right) - \sum_{k=1}^{2^{n-1}-1}\frac{2\theta}{\theta^2-k^2\pi^2}\\
=&\frac{1}{2^n}\sum_{k=1}^{2^{n-1}-1}\left(\cot\left(\frac{\theta+k\pi}{2^n}\right) - \frac{2^n}{\theta+k\pi} + \cot\left(\frac{\theta-k\pi}{2^n}\right) - \frac{2^n}{\theta-k\pi}\right).\\
=&\frac{1}{2^n}\sum_{k=1}^{2^{n-1}-1}\int_{\frac{k\pi-\theta}{2^n}}^{\frac{k\pi+\theta}{2^n}}\left(\frac{1}{t^2} - \csc^2 t\right)\,dt
\end{aligned}$$

Since $0 < -\frac{1}{t^2} + \csc^2 t \leq 1$ for all $t \in \left(\frac{k\pi-\theta}{2^n}, \frac{k\pi+\theta}{2^n}\right) \subset (0, \frac{\pi}{2})$ by Lemma 2.3,

$$0 \leq \frac{1}{2^n} \sum_{k=1}^{2^{n-1}-1} \int_{\frac{k\pi-\theta}{2^n}}^{\frac{k\pi+\theta}{2^n}} \left(-\frac{1}{t^2} + \csc^2 t\right) \, dt \leq \frac{\theta}{2^n}.$$

By Squeeze Theorem, it follows that

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=1}^{2^{n-1}-1} \int_{\frac{k\pi-\theta}{2^n}}^{\frac{k\pi+\theta}{2^n}} \left(\frac{1}{t^2} - \csc^2 t\right) \, dt = 0$$

and consequently,

$$\lim_{n\to\infty} \left\{ \frac{1}{2^n} \sum_{k=1}^{2^{n-1}-1} \left(\cot \frac{\theta + k\pi}{2^n} + \cot \frac{\theta - k\pi}{2^n}\right) - \sum_{k=1}^{2^{n-1}-1} \frac{2\theta}{\theta^2 - k^2\pi^2} \right\} = 0.$$

□

All in all, we have succeeded in showing that

Lemma 3.6. *If $\theta \in (0, \pi)$, the following identity holds:*

$$\cot \theta = \frac{1}{\theta} + \sum_{k=1}^{\infty} \frac{2\theta}{\theta^2 - k^2\pi^2}.$$

To prove Theorem 3.1, we need to extend Lemma 3.6 for $\theta \in \mathbb{R}\backslash\pi\mathbb{Z}$. To prove this assertion, we choose $\alpha \in (0, \pi)$ and $r \in \mathbb{Z}$ so that $\theta = \alpha + r\pi$.

Since $\cot(\theta - r\pi) = \cot \theta$, the result follows from Lemma 3.6 that

$$\begin{aligned}
\cot \theta &= \cot(\theta - r\pi) \\
&= \frac{1}{\theta - r\pi} + \sum_{k=1}^{\infty} \frac{2(\theta - r\pi)}{(\theta - r\pi - k\pi)(\theta - r\pi + k\pi)} \\
&= \frac{1}{\theta - r\pi} + \sum_{k=1}^{\infty} \left(\frac{1}{\theta - r\pi - k\pi} + \frac{1}{\theta - r\pi + k\pi} \right) \\
&= \frac{1}{\theta - r\pi} + \sum_{k=r+1}^{\infty} \left(\frac{1}{\theta - k\pi} + \frac{1}{\theta + (k-r)\pi} \right) + \sum_{k=1}^{r} \frac{1}{\theta + (k-r)\pi} \\
&= \frac{1}{\theta - r\pi} + \sum_{k=r+1}^{\infty} \left(\frac{1}{\theta - k\pi} + \frac{1}{\theta + (k-r)\pi} \right) + \sum_{k=0}^{r-1} \frac{1}{\theta - k\pi} \\
&= \frac{1}{\theta} + \sum_{k=1}^{\infty} \left(\frac{1}{\theta - k\pi} + \frac{1}{\theta + k\pi} \right) \\
&= \frac{1}{\theta} + \sum_{k=1}^{\infty} \frac{2\theta}{\theta^2 - k^2\pi^2}.
\end{aligned}$$

The proof of Theorem 3.1 is now complete.

4 Some applications

Putting $\theta = \pi z$ into (3), $z \notin \mathbb{Z}$, yields immediately:

Corollary 4.1 (Euler's identity)**.** *For $z \in \mathbb{R}\backslash\mathbb{Z}$, it holds that*

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}.$$

An interesting consequence of Theorem 3.1, which is almost immediate, is:

Corollary 4.2 (Neville's identity)**.** *If $\theta \in \mathbb{R}\backslash\pi\mathbb{Z}$, then*

$$\csc^2\theta - \frac{1}{\theta^2} = \sum_{k=1}^{\infty} \left(\frac{1}{(\theta - k\pi)^2} + \frac{1}{(\theta + k\pi)^2} \right). \tag{4}$$

A näive derivation of (4) is via a term-by-term differentiation of (3) since $\frac{2\theta}{\theta^2 - k^2\pi^2} = \frac{1}{\theta - k\pi} + \frac{1}{\theta + k\pi}$. However, this approach involves the unjustified interchange of the differential operator and the infinite sum. We present below a proof to fully justify this interchange of operators without appealing to any convergence theorems.

Proof. Choose $\delta\theta > 0$ so small that $(\theta - \delta\theta, \theta + \delta\theta) \subset (r\pi, (r+1)\pi)$ for some $r \in \mathbb{Z}$. Then for any integer $n > |r|$, we have

$$\begin{aligned}
&\frac{1}{\delta\theta} \left| \sum_{k=1}^{n} \int_{\theta}^{\theta+\delta\theta} \left(\frac{1}{(t - k\pi)^2} + \frac{1}{(t + k\pi)^2} \right) dt - \int_{\theta}^{\theta+\delta\theta} \sum_{k=1}^{\infty} \left(\frac{1}{(t - k\pi)^2} + \frac{1}{(t + k\pi)^2} \right) dt \right| \\
=&\frac{1}{\delta\theta} \left| \int_{\theta}^{\theta+\delta\theta} \sum_{k=n+1}^{\infty} \left(\frac{1}{(t - k\pi)^2} + \frac{1}{(t + k\pi)^2} \right) dt \right| \\
\leq&\frac{1}{\pi^2\delta\theta} \left| \int_{\theta}^{\theta+\delta\theta} \sum_{k=n+1}^{\infty} \left(\frac{1}{(k - r - 1)^2} + \frac{1}{(k + r)^2} \right) dt \right| \\
<& \sum_{k=n+1}^{\infty} \left(\frac{1}{(k - r - 1)^2} + \frac{1}{(k + r)^2} \right) \\
<&\frac{2}{n - 1 - r}.
\end{aligned}$$

Now, Theorem 3.1 asserts that

$$\frac{1}{\delta\theta} \int_{\theta}^{\theta+\delta\theta} \sum_{k=1}^{\infty} \left(\frac{1}{(t - k\pi)^2} + \frac{1}{(t + k\pi)^2} \right) dt = \frac{1}{\delta\theta} \left\{ \frac{1}{\theta + \delta\theta} - \cot(\theta + \delta\theta) - (\frac{1}{\theta} - \cot\theta) \right\}.$$

Taking $\delta\theta \to 0$, the first principle ensures that

$$\left| \sum_{k=1}^{n} \left(\frac{1}{(\theta - k\pi)^2} + \frac{1}{(\theta + k\pi)^2} \right) - \left(-\frac{1}{\theta^2} + \csc^2\theta \right) \right| \leq \frac{2}{n - 1 - r}.$$

Since $\lim_{n\to\infty}\frac{2}{n-1-r}=0$, it follows that

$$\sum_{k=1}^{\infty}\left(\frac{1}{(\theta-k\pi)^2}+\frac{1}{(\theta+k\pi)^2}\right)=\csc^2\theta-\frac{1}{\theta^2}$$

and the proof is now complete. □

Remark 4.3. *The above proof of (4) is much more direct than the original one presented in [7].*

Corollary 4.4 (Euler's formula for $\zeta(2)$)**.**

$$\zeta(2):=\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}.$$

Proof. On one hand, $\lim_{\theta\to 0^+}\sum_{k=1}^{\infty}\left(\frac{1}{(\theta-k\pi)^2}+\frac{1}{(\theta+k\pi)^2}\right)=\frac{2}{\pi^2}\sum_{k=1}\frac{1}{k^2}$. To account for the legitimate interchange of limits, we first observe, for $\theta\in(0,\frac{\pi}{2})$, that

$$\sum_{k=1}^{\infty}\frac{1}{(k\pi+\theta)^2}\le\sum_{k=1}^{\infty}\frac{1}{k^2\pi^2}\le\sum_{k=1}^{\infty}\frac{1}{(k\pi-\theta)^2}.$$

Next, we need to establish that $\sum_{k=1}^{\infty}\left\{\frac{1}{(k\pi-\theta)^2}-\frac{1}{(k\pi+\theta)^2}\right\}\le 16\theta$ for $\theta\in(0,\frac{\pi}{2})$. This derives easily from the following inequalities:

$$\begin{aligned}\sum_{k=1}^{\infty}\left\{\frac{1}{(k\pi-\theta)^2}-\frac{1}{(k\pi+\theta)^2}\right\}&\le\sum_{k=1}^{\infty}\frac{4k\pi\theta}{(k^2\pi^2-\theta^2)^2}\\&\le\sum_{k=1}^{\infty}\frac{4k\theta}{\pi^3(k^2-\frac{1}{4})^2}\\&\le\sum_{k=1}^{\infty}\frac{64k\theta}{(4k^2-1)^2}\le\sum_{k=1}^{\infty}\frac{32\theta}{4k^2-1}\le 16\theta,\end{aligned}$$

where the second-to-last inequality holds since $4k^2-1=(2k-1)(2k+1)\ge 2k$. Thus, $\lim_{\theta\to 0^+}\sum_{k=1}^{\infty}\frac{1}{(k\pi+\theta)^2}=\lim_{\theta\to 0^+}\sum_{k=1}^{\infty}\frac{1}{(k\pi-\theta)^2}=\sum_{k=1}^{\infty}\frac{1}{k^2\pi^2}$. This then implies that $\lim_{\theta\to 0^+}\sum_{k=1}^{\infty}\left(\frac{1}{(\theta-k\pi)^2}+\frac{1}{(\theta+k\pi)^2}\right)=\frac{2}{\pi^2}\sum_{k=1}\frac{1}{k^2}$.

On the other hand, Proposition 2.1 bears upon us to have

$$\frac{(\frac{\theta^3}{3!}-\frac{\theta^5}{5!})(2\theta-\frac{\theta^3}{3!})}{\theta^4}\le\frac{(\theta-\sin\theta)(\theta+\sin\theta)}{\theta^4}\le\frac{\frac{\theta^3}{3!}\cdot 2\theta}{\theta^4}$$

and thus, $\lim_{\theta\to 0^+}(\csc^2\theta-\frac{1}{\theta^2})=\frac{1}{3}$. The desired result then follows. □

5 Concluding remarks

This paper presents a simple and direct proof of the famous Euler's partial fraction representation of the cotangent function. Besides being an intriguing identity in its own right, (3) has seen recent applications in the parametric Euler's iterated sums [1]. We hope to demonstrate, within this small exposition, that even basic calculus techniques can be harnessed to yield nontrivial results that are usually obtained through, otherwise, more difficult means.

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