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A Fourier series approach to calculations related to the evaluation of $\zeta(2n)$

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Abstract

This paper presents a unified approach, based on certain Fourier series, to several interesting calculations related to the evaluation of the Riemann ζ-function at even integral arguments. Our approach reported herein has two main advantages: (i) a lighter mathematical overhead (in fact, no more than freshman calculus, with some very minimal background knowledge in Fourier series), and (ii) an emergence of cleaner proofs of known results.

1 Introduction

In this paper, the Riemann Zeta function $\zeta(s)$ is defined by

$$\zeta(s) := \begin{cases} \sum_{k=1}^{\infty} \frac{1}{k^s} & (\Re(s) > 1) \\ \frac{1}{2^{1-s}} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} & (\Re(s) > 0,\ s \neq 1) \end{cases} \tag{1.1}$$

The usual analytic continuation of ζ allows the value of $\zeta(0)$ to be defined as $-\frac{1}{2}$. Evident from the title of this article, we only need to focus on the Riemann Zeta function ζ restricted on the non-negative integers $\mathbb{N}$. Leonhard Euler studied the ζ-function restricted on the positive integers in 1740, and was the first to answer the Basel problem, i.e., that of presenting the exact value of $\zeta(2)$ in [5]. For a detailed historical account of Euler's solution to the Basel problem, we refer to the excellent work of L. Debnath [4]. The exact values of zeta function at the even integral arguments are given by the formula

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}, \tag{1.2}$$

where B_{2n} is the $2n^{\text{th}}$ Bernoulli number.

There are several recent works revisiting the evaluation of $\zeta(2k)$ using elementary (but not necessarily easy) methods. For example, [7] employs the Chebyshev's polynomials to obtain the infinite product representation for the sine function which in turn yields a proof of $\zeta(2) = \frac{\pi^2}{6}$. However, the method reported therein does not seem to apply to the evaluation of $\zeta(2k)$ for higher values of k. A recent work [6] establishes the value of $\zeta(2)$ using the Neville's identity. A new proof of Equation (1.2) relying on the Taylor series expansion for the tangent function can be found in [3]. No such simple values of the zeta function at odd integer arguments are known to date.

Most elementary methods that deal with calculations related to the ζ-function at nonnegative integers argument are based on special functions and identities, and complex analysis. In this paper, we perform these calculations and derive known results using a comparatively uniform and simple approach via Fourier series. Though the idea that the ζ-function is related to Fourier series has been folkloric, this is only made explicit very recently ([10]). Our method focuses on a specific class of trigonometric series – the associated Clausen functions $\sum_{k=0}^{\infty} \frac{\cos(kx)}{k^{2n}}$ and $\sum_{k=0}^{\infty} \frac{\sin(kx)}{k^{2n+1}}$. Notably, the Fourier cosine series for x^{2m} has been used by N. Robbins in [9] to evaluate $\zeta(2m)$. Our present method differs from [9] in that not only are we able to evaluate ζ at its even integral arguments but also to produce an explicit recurrence relation for ζ (see *loc. cit.* Corollary 3.8).

We organize our paper as follows. In Section 2, we present the preliminary definitions and results. Then, we employ basic calculus techniques to establish some handy identities concerning the associated Clausen functions. In Section 3, a proof of Euler's theorem $\zeta(2) = \frac{\pi^2}{6}$ à la the $\epsilon - N$ definition is given. In Section 4, we establish two infinite series representations concerning the associated Clausen functions. The highlight is a forward substitution method for obtaining the exact values of $\zeta(2n)$, using a simpler recursively-defined sequence α_k in place of the Bernoulli numbers. In particular, we display the calculation of the exact values of $\zeta(2n)$ for $n = 1, 2, \ldots, 5$.

2 Some trigonometric series

2.1 Results concerning sums and series

In this section, we highlight some useful results concerning certain trigonometric series.

Proposition 2.1. *For any $t \in \mathbb{R}$ and $n \in \mathbb{Z}^+$, these identities hold:*

$$\sum_{k=1}^{n} \cos(kt) = \frac{\sin\left(n+\frac{1}{2}\right)t - \sin\frac{1}{2}t}{2\sin\frac{1}{2}t} \tag{2.1}$$

and

$$\sum_{k=1}^{n} \sin(kt) = \frac{\cos\frac{1}{2}t - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} \tag{2.2}$$

Proof. Applying the factor formulae and the method of difference. □

Corollary 2.2. *For any $t \in (0, \pi]$ and $m,\ n \in \mathbb{Z}^+$ where $m < n$,*

$$\left|\sum_{j=m+1}^{n} \sin jt\right| \leq \frac{1}{\sin\frac{1}{2}t}.$$

Proof. By the preceding proposition, we have

$$\left|\sum_{j=m+1}^{n} \sin jt\right| = \left|\frac{\sin\left(\frac{n+m+1}{2}\right)t\sin\left(\frac{n-m}{2}\right)t}{\sin\frac{1}{2}t}\right| \leq \frac{1}{\sin\frac{1}{2}t}.$$

□

Theorem 2.3. *If $x \in (0, \pi)$, then the trigonometric series $\sum_{k=1}^{\infty} \frac{\sin kx}{k}$ converges. Moreover, $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k} = \frac{\pi - x}{2}$.*

Proof. See Lemmata 7.1.5 and 7.1.6 in [8]. □

The following is a specialization of Theorem 7.1.2 of [8]:

Lemma 2.4 (Summation by parts)**.** *Let $(u_k)_{k=0}^{\infty}$ and $(v_k)_{k=0}^{\infty}$ be two sequences of real numbers. Then,*

$$\sum_{k=0}^{n} u_k v_k = v_n \sum_{k=0}^{n} u_j - \sum_{k=0}^{n-1}\left((v_k - v_{k+1})\sum_{j=0}^{k} u_j\right).$$

Lemma 2.5. *Let $t \in (0, \pi]$ and $n \in \mathbb{Z}^+$. Then, the following holds:*

$$\sum_{k=n+1}^{\infty} \frac{\sin(kt)}{k} = \sum_{k=n+1}^{\infty} \frac{1}{k(k+1)} \sum_{j=n+1}^{k} \sin(jt). \tag{2.3}$$

Proof. Let $N \in \mathbb{Z}^+$ be arbitrary.

$$\begin{aligned}
&\left| \sum_{k=n+1}^{N} \frac{\sin(kt)}{k} - \sum_{k=n+1}^{\infty} (\frac{1}{k} - \frac{1}{k+1}) \sum_{j=n+1}^{k} \sin(jt) \right| \\
\leq &\left| \frac{1}{N} \sum_{k=n+1}^{N} \sin(kt) + \sum_{k=n+1}^{N-1} (\frac{1}{k} - \frac{1}{k+1}) \sum_{j=n+1}^{k} \sin(jt) \right. \\
&\left. - \sum_{k=n+1}^{\infty} (\frac{1}{k} - \frac{1}{k+1}) \sum_{j=n+1}^{k} \sin(jt) \right| \quad \text{(by Lemma 2.4)} \\
\leq &\frac{1}{N} \left| \sum_{k=n+1}^{N} \sin(kt) \right| + \sum_{k=N}^{\infty} \left| (\frac{1}{k} - \frac{1}{k+1}) \sum_{j=n+1}^{k} \sin(jt) \right| \\
\leq &\frac{1}{\sin\frac{1}{2}t} \left(\frac{1}{N} + \sum_{k=N}^{\infty} (\frac{1}{k} - \frac{1}{k+1}) \right) \quad \text{(by Corollary 2.2)} \\
\leq &\frac{2}{N \sin\frac{1}{2}t},
\end{aligned}$$

□

3 Evaluation of $\zeta(2n)$

The task of obtaining the exact values of $\zeta(2n)$ begins with the simplest case of $\zeta(2)$, which is also known as the Basel problem. Our approach based on trigonometric series hinges crucially upon the following observation:

Lemma 3.1. *Let $n \in \mathbb{Z}^+$. Then, the n^{th} partial sum of $\zeta(2)$ can be written as*

$$\sum_{k=1}^{n} \frac{1}{k^2} = \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi - t}{2} \right) \sum_{k=1}^{n} \frac{\sin(kt)}{k} \, dt. \tag{3.1}$$

Proof. Follows directly from integration by parts. □

Recalling from Theorem 2.3 that $\sum_{k=1}^{\infty} \frac{\sin(kt)}{k} = \frac{\pi - t}{2}$, one may be tempted to reason naively that by limiting $k \longrightarrow \infty$,

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \int_0^{\pi} \left(\frac{\pi - t}{2} \right) \sum_{k=1}^{\infty} \frac{\sin(kt)}{k} \, dt = \int_0^{\pi} \left(\frac{\pi - t}{2} \right)^2 dt = \frac{\pi^2}{6}.$$

Alas! This reasoning is flawed as $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$ is not uniformly convergent to $\frac{\pi-x}{2}$ on $[0,\pi]$. A more subtle argument in the style of an $\epsilon - N$ proof, as we shall now demonstrate, is required to secure the desired result:

Theorem 3.2 (Euler, 1740)**.**

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{3.2}$$

Proof. By virtue of Lemma 3.1, for every positive integer n, we have

$$\left| \sum_{k=1}^{n} \frac{1}{k^2} - \frac{\pi^2}{6} \right| = \frac{2}{\pi} \left| \int_0^{\pi} \left(\frac{\pi-t}{2} \right) \sum_{k=1}^{n} \frac{\sin(kt)}{k} \, dt - \int_0^{\pi} \left(\frac{\pi-t}{2} \right)^2 dt \right|.$$

This in turn, by the triangle inequality, is bounded above by

$$\underbrace{\left| \int_0^{\frac{1}{n}} \left(\frac{\pi-t}{2} \right) \sum_{k=1}^{n} \frac{\sin(kt)}{k} \, dt - \int_0^{\frac{1}{n}} \left(\frac{\pi-t}{2} \right)^2 dt \right|}_{U} + \underbrace{\left| \int_{\frac{1}{n}}^{\pi} \left(\frac{\pi-t}{2} \right)^2 dt - \int_{\frac{1}{n}}^{\pi} \left(\frac{\pi-t}{2} \right) \sum_{k=1}^{n} \frac{\sin(kt)}{k} \, dt \right|}_{V}.$$

To bound U, consider the inequalities:

$$\begin{aligned} U &\leq \int_0^{\frac{1}{n}} \frac{\pi-t}{2} \sum_{k=1}^{n} \frac{|\sin(kt)|}{k} + \left| \int_0^{\frac{1}{n}} \left(\frac{\pi-t}{2} \right)^2 dt \right| \\ &= \left(\frac{\pi}{2n} \sum_{k=1}^{n} \frac{1}{k} \right) + \frac{\pi^2}{4n}. \end{aligned}$$

To bound V, first exploit Theorem 2.3 and Lemma 2.5 to rewrite the difference of the integrals as

$$\begin{aligned} &\int_{\frac{1}{n}}^{\pi} \left(\frac{\pi-t}{2} \right) \left[\frac{\pi-t}{2} - \sum_{k=1}^{n} \frac{\sin(kt)}{k} \right] dt \\ &= \int_{\frac{1}{n}}^{\pi} \left(\frac{\pi-t}{2} \right) \sum_{k=n+1}^{\infty} \frac{\sin(kt)}{k} \, dt \\ &= \int_{\frac{1}{n}}^{\pi} \left(\frac{\pi-t}{2} \right) \sum_{k=n+1}^{\infty} \frac{1}{k(k+1)} \sum_{j=k+1}^{k} \sin(jt) \, dt, \end{aligned}$$

and then use Corollary 2.2 to bound this above by

$$\frac{\pi}{2} \int_{\frac{1}{n}}^{\pi} \frac{1}{\sin \frac{1}{2} t} \sum_{k=n+1}^{\infty} \frac{1}{k(k+1)} \, dt \leq \frac{\pi^2}{2(n+1)} (\ln \pi + \ln n).$$

So, all in all, we have

$$\begin{aligned}\left|\sum_{k=1}^{n}\frac{1}{k^2}-\frac{\pi^2}{6}\right| \leq & \frac{\pi}{2n}\sum_{k=1}^{n}\frac{1}{k}+\frac{\pi^2}{4n}+\frac{\pi^2}{2(n+1)}(\ln\pi+\ln n)\\ \leq & \frac{2}{n}\sum_{k=1}^{n}\frac{2}{\sqrt{k}+\sqrt{k-1}}+\frac{3}{n}+\frac{4^2}{2(n+1)}(\ln 4+2\sqrt{n})\\ \leq & \frac{4}{\sqrt{n}}+\frac{3}{n}+\frac{16}{n+1}+\frac{16}{\sqrt{n}}\leq\frac{39}{\sqrt{n}}.\end{aligned}$$

□

Having gained success on the exact evaluation of $\zeta(2)$ using trigonometric series such as $\sum_{k=1}^{\infty}\frac{\sin(kx)}{k}$, it is natural to ask if similar series can be considered in the exact evaluation of $\zeta(2n)$ for positive integers n. In the ensuing development, we carry out this plan by considering the following two trigonometric series:

$$\mathrm{C}_n(x) := \sum_{k=1}^{\infty}\frac{\cos(kx)}{k^{2n}} \text{ and } \mathrm{S}_n(x) := \sum_{k=1}^{\infty}\frac{\sin(kx)}{k^{2n+1}},\ x\in[0,\pi],\ n\in\mathbb{Z}^+$$

They are sometimes referred to as the *associated Clausen functions* in the existing literature (see, for example, p.201 of [2]). We record here an elementary property concerning these series:

Proposition 3.3. *Let $n\in\mathbb{Z}^+$. Then, the series*

$$\sum_{k=1}^{\infty}\frac{\cos(kx)}{k^{2n}} \text{ and } \sum_{k=1}^{\infty}\frac{\sin(kx)}{k^{2n+1}},\quad x\in[0,2\pi]$$

converge absolutely (and hence uniformly) on $[0,2\pi]$.

Our inductive approach bears upon us to begin with $n=1$.

Theorem 3.4. *If $x\in[0,2\pi]$, then $\sum_{k=1}^{\infty}\frac{\cos kx}{k^2}$ converges and*

$$\mathrm{C}_1(x)=\sum_{k=1}^{\infty}\frac{\cos(kx)}{k^2}=-\frac{\zeta(2)}{2}+\frac{(\pi-x)^2}{4}. \tag{3.3}$$

Proof. In view of Theorem 3.2, we may assume that $x\in(0,2\pi)$. Since for any positive integer n, it holds that

$$\begin{aligned}\left|\int_x^{\pi}\sum_{k=1}^{n}\frac{\sin(kt)}{k}\,dt-\int_x^{\pi}\frac{\pi-t}{2}\,dt\right| =& \left|\int_x^{\pi}\sum_{k=n+1}^{\infty}\frac{\sin(kt)}{k}\,dt\right|\\ =& \left|\int_x^{\pi}\sum_{k=n+1}^{\infty}\frac{1}{k(k+1)}\sum_{j=n+1}^{k}\sin(jt)\,dt\right|\\ \leq& \frac{1}{n+1}\left|\int_x^{\pi}\frac{1}{\sin\frac{1}{2}t}\,dt\right|,\end{aligned}$$

it follows that

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}-\sum_{k=1}^{\infty}\frac{\cos(kx)}{k^2}=\int_x^{\pi}\sum_{k=1}^{\infty}\frac{\sin(kt)}{k}\,dt=\int_0^{\pi}\frac{\pi-t}{2}\,dt=\frac{(\pi-x)^2}{4}.$$

Finally, because $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}=-\frac{\zeta(2)}{2}$, we obtain the desired result. □

Remark 3.5. Notice that the proof technique used in the preceding theorem cannot be cheaply extended to account for the convergence at $x=0$ since the bounding technique involves the function $\frac{1}{\sin\frac{1}{2}t}$ which has a singularity at $t=0$.

Theorem 3.6. *If $x\in[0,2\pi]$, then $\sum_{k=1}^{\infty}\frac{\sin kx}{k^3}$ converges and*

$$S_1(x):=\sum_{k=1}^{\infty}\frac{\sin(kx)}{k^3}=\frac{\zeta(2)}{2}(\pi-x)-\frac{(\pi-x)^3}{12}. \tag{3.4}$$

Proof. The result follows immediately from

$$\left|\int_x^{\pi}\left(\sum_{k=1}^{n}\frac{\cos kt}{k^2}-\sum_{k=1}^{\infty}\frac{\cos kt}{k^2}\right)dt\right|\to 0 \text{ as } n\to\infty.$$

and Theorem 3.4. □

To find the exact value of $\zeta(4)$, one could have proceeded by using the result that

$$S_1(x)=\frac{\zeta(2)}{2}(\pi-x)-\frac{(\pi-x)^3}{12}$$

and the following fact (analogous to Lemma 3.1) that

$$\sum_{k=1}^{n}\frac{1}{k^4}=\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi-t}{2}\right)\sum_{k=1}^{n}\frac{\sin(kt)}{k^3}\,dt$$

and arguing along the line of reasoning used in Theorem 3.2. However, we choose not to do so. Instead, we opt for a more elegant method that would derive the evaluation of ζ at all positive even integer arguments. This necessarily calls for generalizations of Theorems 3.4 and 3.6, which appear as follows.

Theorem 3.7. *For each positive integer n and $x\in[0,\pi]$, the following hold:*

$$\sum_{k=1}^{\infty}\frac{\cos(kx)}{k^{2n}} = \sum_{r=0}^{n}(-1)^{r+1}\cdot\left(1-\frac{1}{2^{2n-2r-1}}\right)\cdot\zeta(2n-2r)\cdot\frac{(\pi-x)^{2r}}{(2r)!} \tag{3.5}$$

$$\sum_{k=1}^{\infty}\frac{\sin(kx)}{k^{2n+1}} = \sum_{r=0}^{n}(-1)^{r}\cdot\left(1-\frac{1}{2^{2n-2r-1}}\right)\cdot\zeta(2n-2r)\cdot\frac{(\pi-x)^{2r+1}}{(2r+1)!} \tag{3.6}$$

Proof. We proceed by induction as promised earlier. The base cases of $n = 1$ are just Theorems 3.4 and 3.6. Assuming that the statements hold for n, we must show that they hold for $n + 1$. Since Theorem 3.3 guarantees that

$$\int_x^\pi \sum_{k=1}^\infty -\frac{\sin(kt)}{k^{2n+1}}\, dt = -\sum_{k=1}^\infty \int_x^\pi \frac{\sin(kt)}{k^{2n+1}}\, dt = \sum_{k=1}^\infty \frac{\cos(k\pi)}{k^{2n+2}} - \sum_{k=1}^\infty \frac{\cos(kx)}{k^{2n+2}},$$

it follows that

$$\begin{aligned}\sum_{k=1}^\infty \frac{\cos(kx)}{k^{2n+2}} &= \sum_{k=1}^\infty \frac{(-1)^k}{k^{2n+2}} + \int_x^\pi \sum_{k=1}^\infty \frac{\sin(kt)}{k^{2n+1}}\, dt \\ &= -\left(1 - \frac{1}{2^{2n+1}}\right)\zeta(2n+2) + \int_x^\pi \sum_{k=1}^\infty \frac{\sin(kt)}{k^{2n+1}}\, dt.\end{aligned}$$

Invoking the induction hypothesis, one has

$$\begin{aligned}\sum_{k=1}^\infty \frac{\cos(kx)}{k^{2n+2}} = &-\left(1 - \frac{1}{2^{2n+1}}\right)\zeta(2n+2) \\ &+ \int_x^\pi \sum_{r=0}^{n}(-1)^r \cdot \left(1 - \frac{1}{2^{2n-2r-1}}\right) \cdot \zeta(2n-2r) \cdot \frac{(\pi - t)^{2r+1}}{(2r+1)!}\, dt \\ = &\sum_{r=0}^{n+1}(-1)^{r+1} \cdot \left(1 - \frac{1}{2^{2(n+1)-2r-1}}\right) \cdot \zeta(2(n+1)-2r) \cdot \frac{(\pi - x)^{2r}}{(2r)!}.\end{aligned}$$

Similar reasoning applies for the sine counterpart. □

Corollary 3.8. *Let $n \in \mathbb{Z}^+$. Then, the following hold:*

1. $\left(2 - \frac{1}{2^{2n-1}}\right) \cdot \zeta(2n) + \sum_{r=1}^{n}(-1)^r \left(1 - \frac{1}{2^{2n-2r-1}}\right) \cdot \zeta(2n-2r) \cdot \frac{\pi^{2r}}{(2r)!} = 0.$
2. $\sum_{r=0}^{n}(-1)^r \left(1 - \frac{1}{2^{2n-2r-1}}\right) \cdot \zeta(2n-2r) \cdot \frac{\pi^{2r}}{(2r+1)!} = 0.$

Proof.
1. Put $x = 0$ in the cosine series of Theorem 3.7.
2. Put $x = 0$ in the sine series of Theorem 3.7.

□

From Corollary 3.8(2), substituting values of $n = 1, 2, \dots$ one obtains the linear system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = (a_{ij})$ is defined by

$$a_{ij} = \begin{cases}(-1)^{i+j}\left(1 - \frac{1}{2^{2j-3}}\right)\frac{\pi^{2i-2j}}{(2i-1)!} & \text{if } i < j; \\ 0 & \text{otherwise.}\end{cases}$$

and $\mathbf{x} = (\zeta(0)\ \zeta(2)\ \cdots\ \zeta(2n))^T$ and $\mathbf{b} = (\frac{1}{2}\ 0\ \cdots\ 0)^T$. Via forward substitution, one easily obtains an expression for $\zeta(2n)$ as a rational multiple of π^{2n} without the use of Bernoulli numbers.

Theorem 3.9. *Let $n \in \mathbb{N}$. Then, one has $\zeta(2n) = \frac{2^{2n-2}\pi^{2n}}{2^{2n-1}-1}\alpha_n$, where*

$$\alpha_0 = 1 \text{ and } \alpha_n = \sum_{r=1}^{n}(-1)^{r+1}\frac{\alpha_{n-r}}{(2r+1)!}, \quad n \in \mathbb{Z}^+.$$

Proof. The proof proceeds by induction on n.
For the base case of $n = 0$, because $\frac{2^{2(0)-2}\pi^0}{2^{2(0)-1}-1}\alpha_0 = -\frac{1}{2} = \zeta(0)$, the statement holds. For the inductive step, we assume that

$$\zeta(2k) = \frac{2^{2k-2}\pi^{2k}}{2^{2k-1}-1}\alpha_k, \qquad 1 \leq k < n.$$

The proof at the inductive step now proceeds as follows:

$$\begin{aligned}
\zeta(2n) &= \left(1 - \frac{1}{2^{2n-1}}\right)^{-1}\sum_{r=1}^{n}(-1)^{r+1}\left(1 - \frac{1}{2^{2n-2r-1}}\right)\zeta(2n-2r)\cdot\frac{\pi^{2r}}{(2r+1)!} \\
&= \frac{2^{2p-1}}{2^{2n-1}-1}\sum_{r=1}^{n}(-1)^{r+1}\frac{2^{2n-2r-1}-1}{2^{2n-2r-1}}\cdot\frac{2^{2k-2r-2}\pi^{2n-2r}}{2^{2n-2r-1}-1}\alpha_{n-r}\cdot\frac{\pi^{2r}}{(2r+1)!} \\
&= \frac{2^{2n-2}\pi^{2n}}{2^{2n-1}-1}\sum_{r=1}^{n}(-1)^{r+1}\frac{\alpha_{n-r}}{(2r+1)!}. \\
&= \frac{2^{2n-2}\pi^{2n}}{2^{2n-1}-1}\alpha_n.
\end{aligned}$$

□

Remark 3.10. A similar result expressing $\zeta(2k)$ as $a_k\pi^{2k}$ where a_k are given recursively by $\sum_{j=1}^{m}\frac{(-1)^j a_j}{(2m+1-2j)!} = \frac{m}{(2m+1)!}$ can be found in [1]. Our recurrence relation in Theorem 3.9 used to calculate the α_k's is easier to apply, compared to Chen's recurrence formula.

As is well known that $\zeta(2n) = \frac{(-1)^{n-1}}{2(2n)!}B_{2n}(2\pi)^{2n}$, where B_{2n} is the $2n^{\text{th}}$ Bernoulli number, it must be that $\frac{(-1)^{n-1}}{2(2n)!}B_{2n}(2\pi)^{2n} = \frac{2^{2n-2}\pi^{2n}}{2^{2n-1}-1}\alpha_n$, which implies that

Corollary 3.11. *For any positive integer n, $B_{2n} = \frac{(-1)^{n-1}(2n)!}{2^{2n}-2}\alpha_n$.*

Compared to the form involving Bernoulli numbers, the formula for computing the exact values of $\zeta(2n)$ given in Theorem 3.9 is easier to use. For instance, a direct computation yields

$$\alpha_1 = \frac{1}{6},\ \alpha_2 = \frac{7}{360},\ \alpha_3 = \frac{31}{15120},\ \alpha_4 = \frac{127}{604800},\ \alpha_5 = \frac{73}{3421440}.$$

This then returns:

$$\begin{aligned}
&\zeta(2) = \tfrac{2^{2(1)-2}\pi^2}{2^{2(1)-1}-1}\alpha_1 = \tfrac{1}{6}\pi^2, && \zeta(4) = \tfrac{2^{2(2)-2}\pi^4}{2^{2(2)-1}-1}\alpha_2 = \tfrac{1}{90}\pi^4, \\
&\zeta(6) = \tfrac{2^{2(3)-2}\pi^6}{2^{2(3)-1}-1}\alpha_3 = \tfrac{1}{945}\pi^6, && \zeta(8) = \tfrac{2^{2(4)-2}\pi^8}{2^{2(4)-1}-1}\alpha_4 = \tfrac{1}{9450}\pi^8, \\
&\zeta(10) = \tfrac{2^{2(5)-2}\pi^{10}}{2^{2(5)-1}-1}\alpha_5 = \tfrac{1}{93555}\pi^{10}.
\end{aligned}$$

4 Concluding remarks

In this paper, we have presented a unified approach to evaluate $\zeta(2k)$ based on elementary calculus techniques applied to associated Clausen functions. In the course of proving the famous formula (1.2) for $\zeta(2k)$, we produced two recurrence formulae for ζ (c.f. Corollary 3.8). Our Fourier-series methodology has been further developed to yield representations of $\zeta(2k+1)$ as an infinite series involving $\zeta(2j)$'s, fast-convergent series representations for $\zeta(2k+1)$, log-sine integral formulae and so on.

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